Probabilistic Symmetry and Network Models
Lecture 1

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0. Introduction

If you had asked a probabilist in 1970 what was known about exchangeability, you would likely have received the answer "There's de Finetti's theorem: what else is there to say?" The purpose of these notes is to dispel this (still prevalent) attitude by presenting, in Parts II-IV, a variety of mostly post-1970 results relating to exchangeability. The selection of

From Aldous (1983). *Exchangeability and Related Topics*.

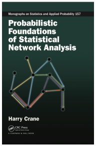
Also: O. Kallenberg. (2006). *Probabilistic Symmetries and Invariance Principles*.

Today: Mostly post-2015 results relating to exchangeability.

Plan

- **Lecture 1: Basic symmetries and network sampling.**
  - H. Crane. (2018). *Probabilistic Foundations of Statistical Network Analysis*.
  - H. Crane and W. Dempsey. (2018). Edge exchangeable models for interaction networks. *Journal of the American Statistical Association*.
  - H. Crane and W. Dempsey. (2019). Relational exchangeability. *Journal of Applied Probability*.
  - H. Crane and H. Towsner. (2018). Relatively exchangeable structures. *Journal of Symbolic Logic*.

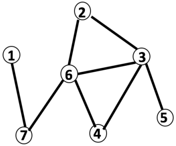
- Lecture 2: Dynamic network models.
  - H. Crane. (2015). Time-varying network models. *Bernoulli*, **21**(3):1670–1696.
  - H. Crane. (2016). Dynamic random networks and their graph limits. *Annals of Applied Probability*.
  - H. Crane. (2017). Exchangeable graph-valued Feller processes. *Probability Theory and Related Fields*.
  - H. Crane. (2018). Combinatorial Lévy processes. *Annals of Applied Probability*.
  - H. Crane and H. Towsner. (2019+). The structure of combinatorial Markov processes.



Book website: `http://www.harrycrane.com/networks.html`

Notation and Terminology

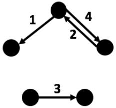
- **Network**: Abstract, non-mathematical concept. A structure of interconnected and/or interacting entities.
- **Graph**: a pair (V, E) consisting of sets V (vertices) and $E \subset V \times V$ (vertices).



  - Encoded as $\{0, 1\}$-valued adjacency array $\mathbf{y} = (y_{ij})_{i,j \in V}$ with

$$y_{ij} = \begin{cases} 1, & (i, j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

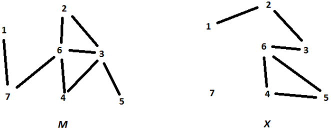
- **Edge-labeled graph**: Equivalence class of structures formed out of edge sequences (formal definition later).



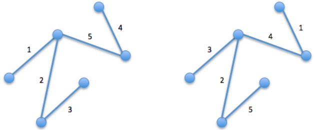
Variations of exchangeability

1. **Exchangeability** (conventional): $\mathbf{X} =_{\mathcal{D}} \mathbf{X}^{\sigma} = (X_{\sigma(i)\sigma(j)})_{i,j\geq 1}$ for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$.

2. **Relative exchangeability**: M represents heterogeneity of a population and $\mathbf{X}$ is exchangeable **relative to** the symmetries of M.



3. **Relational exchangeability**: Exchangeability with respect to relabeling relations (species, edges, paths, networks, etc.)



Exchangeable random graphs

Vertex relabeling

- **Data**: $\mathbf{Y} = (Y_{ij})_{i,j \in V} \in \{0,1\}^{V \times V}$.
- **Symmetries**: Any permutation $\sigma : V \to V$ determines a **relabeling** map

$$\mathbf{y} \mapsto \mathbf{y}^{\sigma} := (y_{\sigma(i)\sigma(j)})_{i,j \in V}.$$

Graph on the right obtained by relabeling graph on left with
$\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 1$, $\sigma(4) = 7$, $\sigma(5) = 3$, $\sigma(6) = 5$, $\sigma(7) = 6$.

Definition ((Vertex) exchangeability)

A random graph $\mathbf{Y} = (Y_{ij})_{i,j \in V}$ *is* (vertex) exchangeable *if*

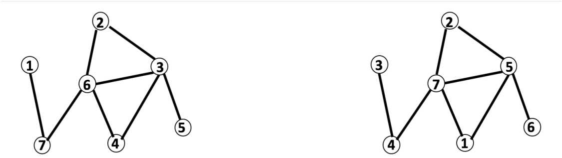
$$\mathbf{Y}^{\sigma} =_{\mathcal{D}} \mathbf{Y} \quad \textit{for all permutations } \sigma : V \to V.$$

Equivalently,

$$\mathbb{P}(\mathbf{Y} \in A^{\sigma}) = \mathbb{P}(\mathbf{Y} \in A) \quad \textit{for all permutations } \sigma : V \to V,$$

where $A^{\sigma} = \{\mathbf{y}^{\sigma} : \mathbf{y} \in A\}$ *obtained by relabeling all elements of* A *according to* σ*.*

- A vertex exchangeable distribution assigns equal probability to isomorphic graphs.



de Finetti's theorem

A sequence $\mathbf{X} = (X_1, X_2, \ldots)$ is **exchangeable** if

$$\mathbf{X}^{\sigma} = (X_{\sigma(1)}, X_{\sigma(2)}, \ldots) =_{\mathcal{D}} \mathbf{X} \quad \text{for all permutations } \sigma : \mathbb{N} \to \mathbb{N}.$$

Theorem (de Finetti)

Let $\mathbf{X} = (X_1, X_2, \ldots)$ *be a countable, exchangeable* $\{0, 1\}$*-valued sequence. Then there exists a unique probability measure* μ *on* $[0, 1]$ *such that the finite-dimensional distributions of* $\mathbf{X}$ *are given by*

$$Pr((X_1, \ldots, X_n) = (x_1, \ldots, x_n)) = \int_0^1 p^{\sum_i x_i}(1-p)^{n-\sum_i x_i}\mu(dp).$$

- Intuition:
  - Pick a coin with a random heads-probability (according to μ).
  - Toss the coin repeatedly to generate $\mathbf{X}$.
- An exchangeable sequence is a mixture of i.i.d. sequences.
- Analogous theorem holds for countably exchangeable sequences in any nice enough probability space.

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de Finetti's theorem: alternative representation

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Let $\mathbf{X} = (X_1, X_2, \ldots)$ be an exchangeable $\{0,1\}$-valued sequence. Then there exists a measurable function $\phi : [0,1] \times [0,1] \to \{0,1\}$ such that $\mathbf{X} =_{\mathcal{D}} \mathbf{X}^* = (X_1^*, X_2^*, \ldots)$, where

$$X_j^* = \phi(U_\emptyset, U_{\{j\}}), \quad j \geq 1,$$

for $U_\emptyset$, $(U_{\{i\}})_{i \geq 1}$ i.i.d. Uniform$[0,1]$.

- Pick a coin with a random heads-probability (according to μ).
- Toss the coin repeatedly to generate $\mathbf{X}$.

- Shared dependence on $U_\emptyset$ is the only source of dependence among variables.
- Fix $U_0 = \alpha$, then $X_j^* = \phi(\alpha, U_{\{j\}})$ is i.i.d.

Exchangeable sequence is conditionally i.i.d.

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Exchangeable random graphs

A random array $\mathbf{Y} = (Y_{i,j})_{i,j \geq 1}$ is **exchangeable** if

$$\mathbf{Y}^{\sigma} = (Y_{\sigma(i)\sigma(j)})_{i,j\geq 1} =_{\mathcal{D}} \mathbf{Y} \quad \text{for all permutations } \sigma : \mathbb{N} \to \mathbb{N}.$$

Theorem (Aldous–Hoover–Kallenberg)

Let $\mathbf{Y}$ be the adjacency array of an exchangeable random graph with vertex set $\mathbb{N}$. Then there exists a measurable function $\phi : [0,1]^4 \to \{0,1\}$ such that $\mathbf{Y} =_{\mathcal{D}} \mathbf{Y}^* = (Y^*_{ij})_{i,j\in\mathbb{N}}$, where

$$Y^*_{ij} = \phi(U_{\emptyset}, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad i,j \geq 1,$$

for $U_{\emptyset}$, $(U_{\{i\}})_{i\geq 1}$, $(U_{\{i,j\}})_{j\geq i\geq 1}$ i.i.d. Uniform$[0,1]$.

Decomposes structure of exchangeable random graph:

- Global effect: $U_{\emptyset}$
- Vertex effects: $U_{\{i\}}$, $U_{\{j\}}$
- Edge effects: $U_{\{i,j\}}$

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Decomposes structure of exchangeable random graph:

- Global effect: $U_{\emptyset}$
- Vertex effects: $U_{\{i\}}$, $U_{\{j\}}$ (shared by all edges involving given vertex)
- Edge effects: $U_{\{i,j\}}$

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Decomposes structure of exchangeable random graph:

- Global effect: $U_{\emptyset}$
- Vertex effects: $U_{\{i\}}, U_{\{j\}}$
- Edge effects: $U_{\{i,j\}}$ (only for edge between i and j)

Dissociated random graphs

Definition (Dissociated array)

A random array $\mathbf{Y} = (Y_{ij})_{i,j\geq 1}$ *is* **dissociated** *if*

$$\mathbf{Y}|_S \text{ and } \mathbf{Y}|_T \text{ are independent for all } S, T \subseteq \mathbb{N} \text{ such that } S \cap T = \emptyset.$$

Fix $U_\emptyset = \alpha$ in Aldous–Hoover: for $S \cap T = \emptyset$:

- $\mathbf{Y}^*|_S = (\phi(\alpha, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}))_{i,j\geq S}$ depends on $U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}$ indexed by S.
- $\mathbf{Y}^*|_T = (\phi(\alpha, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}))_{i,j\geq T}$ depends on $U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}$ indexed by T.

Every exchangeable random graph is a mixture of dissociated, exchangeable graphs.

Example:

- Erdős–Rényi model: all edges are i.i.d. (ϕ depends only on last argument.)
- Graphon models: let $g : [0,1] \times [0,1] \to [0,1]$, let $U_1, U_2, \ldots$ i.i.d. Uniform$[0,1]$ and let
$$\mathbb{P}(Y_{ij} = 1 \mid U_i, U_j) = g(U_i, U_j), \quad i, j \geq 1.$$

Graph limits

Let $G = (G_{ij})_{i,j\in\mathbb{N}}$ be a countable graph and $F = (F_{ij})_{1\leq i,j\leq m}$ be a graph with vertex set $[m] = \{1, \ldots, m\}$.

- For each $n \geq 1$, define

$$t_n(F, G) = \frac{1}{n^{\downarrow m}} \sum_{\text{injections } \psi:[m]\to[n]} \mathbf{1}(G^{\psi} = F).$$

- The **homomorphism density of F in G** is the limit

$$t(F, G) = \lim_{n\to\infty} t_n(F, G) \quad \text{if the limit exists.}$$

- G possesses a **graph limit** if $t(F, G)$ exists for all finite F, for all $m \geq 1$.

Corollary

Graph limits $\longleftrightarrow$ exchangeable, dissociated probability measures on countable graphs.

Immediate implications:

(i) Dense structure: Exchangeable random graph $\Longrightarrow$ dense or empty w.p. 1.

(ii) Representative sampling: normalizing constant $1/n^{\downarrow m}$ interpreted as assigning equal probability (uniform distribution) on all ψ-sampling maps.

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Open Problem

Define and study an interesting notion of asymptotics for sparse/complex networks.

Relative exchangeability

Definition (Relative exchangeability)

Invariance with respect to the symmetries of another structure.

Population $\mathbb{N} = \{1, 2, \ldots\}$ divides into two classes, e.g., male and female.

- Define $C = (C_1, C_2, \ldots)$ by

$$C_i = \begin{cases} 1, & i \text{ is male,} \\ 0, & \text{otherwise.} \end{cases}$$

- $(X_1, X_2, \ldots)$ is relatively exchangeable with respect to C, i.e., $\mathbf{X}^\sigma =_{\mathcal{D}} \mathbf{X}$ for permutations $\sigma : \mathbb{N} \to \mathbb{N}$ that fix C.

C	1	1	0	1	0	0	1
$\mathbf{X}$	X_1	X_2	X_3	X_4	X_5	X_6	X_7
$\mathbf{X}'$	X_2	X_4	X_3	X_1	X_6	X_5	X_7

$$\mathbf{X} =_{\mathcal{D}} \mathbf{X}'$$

Relatively exchangeable sequences

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Theorem

Let $C = (C_1, C_2, \ldots)$ be $[k]$-valued sequence and $\mathbf{X} = (X_1, X_2, \ldots)$ be relatively exchangeable with respect to C. Then there exists a measurable $\phi : [k] \times [0,1]^2 \to \{0,1\}$ such that $\mathbf{X} =_{\mathcal{D}} \mathbf{X}^* = (X_i^*)_{i\geq 1}$ with*

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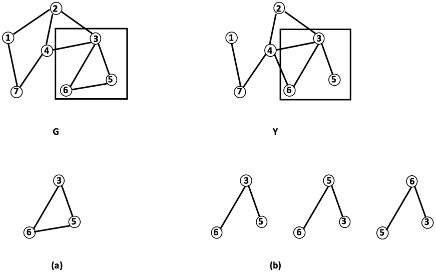
$\mathbf{Y}$ is relatively exchangeable with respect to G if, for all $S \subseteq \mathbb{N}$, $\mathbf{Y}|_S^\sigma =_{\mathcal{D}} \mathbf{Y}|_S$ for all automorphisms σ of $G|_S$.



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Theorem (C. 2017)

Let $G = (\mathbb{N}, E)$ *be an undirected graph* and* $\mathbf{Y}$ *be relatively exchangeable with respect to G. There exists* $\phi : \{0,1\} \times [0,1]^4 \to \{0,1\}$ *such that* $\mathbf{Y} =_{\mathcal{D}} \mathbf{Y}^* = (Y^*_{ij})_{i,j\geq 1}$ *with*

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$$G_{ij} = \begin{cases} 1, & (i,j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

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- exchangeable (Aldous–Hoover) component

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General case

$\mathfrak{M}$: general combinatorial structure

Y is $\mathfrak{M}$-exchangeable (exchangeable relative to $\mathfrak{M}$):

Sequence: $\mathfrak{M} = (M_1, M_2, \ldots) \in \{0,1\}^{\mathbb{N}}$,

$$Y_i = \phi(M_i, U_\emptyset, U_{\{i\}}), \quad i \geq 1.$$

Graph: $\mathfrak{M} = (M_{ij})_{i,j\geq 1} \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$,

$$Y_{ij} = \phi(M_{ij}, U_\emptyset, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad i,j \geq 1.$$

Lack of interference: *Both exhibit strong local dependence on* $\mathfrak{M}$*.*

- Does this lack of interference hold in general? No.
- What properties must $\mathfrak{M}$ satisfy to get the representation?

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Notation

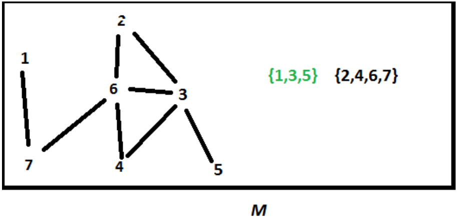
General setting:

- **signature**: $\mathcal{L} = \{i_1, \ldots, i_r\}$ with $1 \leq i_1 \leq \cdots \leq i_r$.
- **$\mathcal{L}$-structure**: $\mathfrak{M} = (\mathfrak{M}^1, \ldots, \mathfrak{M}^r)$ with each $\mathfrak{M}^j$ a symmetric i_j-ary relation $\mathfrak{M}^j \subseteq \mathbb{N}^{i_j}$.
- **adjacency array**: $\mathfrak{M} = (\mathfrak{M}^1, \ldots, \mathfrak{M}^r)$ corresponds to a collection of $\{0, 1\}$-valued arrays $\mathfrak{M}^j = (\mathfrak{M}^j_s, s \in \mathbb{N}^{i_j})$ with

$$\mathfrak{M}^j_s = 1 \quad \Longleftrightarrow \quad s \in \mathfrak{M}^j.$$

Example:

- $\mathcal{L} = \{1, 2\}$: Graph with colored vertices



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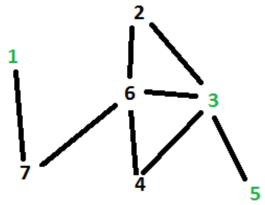

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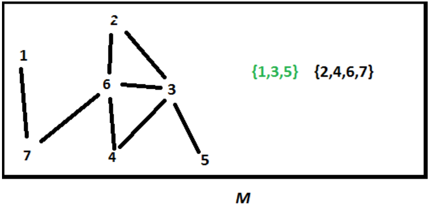
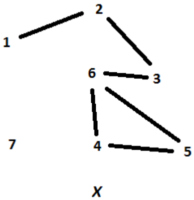
- random structure: $\mathbf{Y} = (Y_{ij})_{i,j \geq 1}$.

Definition

$\mathbf{Y}$ *is* **relatively exchangeable with respect to** $\mathfrak{M}$ *if* $\mathbf{Y}\,|^{\sigma}_{S} =_{\mathcal{D}} \mathbf{Y}\,|_{S}$ *for all permutations* $\sigma : S \to S$ *such that* $\mathfrak{M}|^{\sigma}_{S} = \mathfrak{M}|_{S}$.





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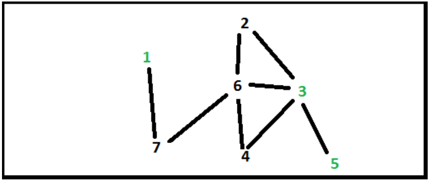
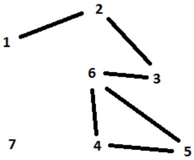


M



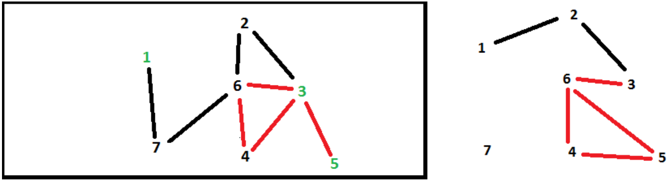
X

Relative exchangeability

- population structure: $\mathfrak{M} = (\mathfrak{M}^1, \ldots, \mathfrak{M}^r)$ with each $\mathfrak{M}^j = (\mathfrak{M}^j_s, s \in \mathbb{N}^{i_j})$ for $i_j \geq 1$.
- random structure: $\mathbf{Y} = (Y_{ij})_{i,j\geq 1}$.

Definition

$\mathbf{Y}$ *is* **relatively exchangeable with respect to** $\mathfrak{M}$ *if* $\mathbf{Y}\,|_S^\sigma =_{\mathcal{D}} \mathbf{Y}\,|_S$ *for all permutations* $\sigma : S \to S$ *such that* $\mathfrak{M}|_S^\sigma = \mathfrak{M}|_S$.



M $\qquad$ X

General representation

Theorem (C.-Towsner, 2018)

Let $\mathbf{Y} = (Y_{ij})_{i,j\geq 1}$ be relatively exchangeable with respect to $\mathfrak{M} = (\mathfrak{M}^1, \ldots, \mathfrak{M}^r)$. Then there exists ϕ such that $\mathbf{Y} =_{\mathcal{D}} \mathbf{Y}^*$ with

$$Y_{ij} = \phi(\mathfrak{M}|_{\{i,j\}}, U_\emptyset, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad (i,j) \in \mathbb{N}, \tag{1}$$

for $U_\emptyset$, $(U_{\{i\}})_{i\geq 1}$, $(U_{\{i,j\}})_{j\geq i\geq 1}$ i.i.d. Uniform$[0,1]$ and $\mathfrak{M}|_S := (\mathfrak{M}^1|_S, \ldots, \mathfrak{M}^r|_S)$.

Representation in (3) holds only under strong condition on $\mathfrak{M}$:

- **ultrahomogeneous**: every embedding $\mathfrak{N} \to \mathfrak{M}$ extends to a automorphism of $\mathfrak{M}$.
- ***n*-disjoint amalgamation (*n*-DAP)**: Let K (set of finite structures) be closed under isomorphism. For every $(\mathfrak{S}_i)_{1\leq i\leq n}$ satisfying
  - $\mathfrak{S}_i \in K$,
  - $|\mathfrak{S}_i| = [n] \setminus \{i\}$,
  - and $\mathfrak{S}_i|_{[n]\setminus\{i,j\}} = \mathfrak{S}_j|_{[n]\setminus\{i,j\}}$ for all $1 \leq i,j \leq n$,

  there exists $\mathfrak{S} \in K$ with $|\mathfrak{S}| = n$ such that $\mathfrak{S}|_{[n]\setminus\{i\}} = \mathfrak{S}_i$ for all $1 \leq i \leq n$.

- 3-DAP (sets): $\mathfrak{S}_i \in \{\{1,3\},\{1,2\},\{2,3\}\}$ extends to $\{1,2,3\}$.
- 3-DAP fails (partitions): $\mathfrak{S}_i \in \{1/3, 12, 2/3\}$ cannot be extended to a partition of [3].

Relational/Edge exchangeability

Species sampling

Sample animals and record their species

$$\underbrace{\text{bear},}_{X_1}\quad \underbrace{\text{deer},}_{X_2}\quad \underbrace{\text{bear},}_{X_3}\quad \underbrace{\text{wolf},}_{X_4}\quad \ldots$$

- Element-labeled sequence: $X_1, X_2, X_3, X_4, \ldots$

- Relationally-labeled structure: $\sim_{\mathbf{X}} \equiv \{1,3\}, \{2\}, \{4\}, \ldots$

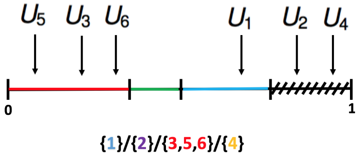
Invariance:

- $(X_1, X_2, \ldots) =_{\mathcal{D}} (X_{\sigma(1)}, X_{\sigma(2)}, \ldots)$: observed species representative of all species.

- $\sim_{\mathbf{X}^\sigma} =_{\mathcal{D}} \sim_{\mathbf{X}}$: *relation* among observed species is representative of the relation of all species.

Kingman's paintbox representation

- **Partition of** $[n]$: $\pi = B_1/B_2/\cdots/B_k$ with nonempty, disjoint subsets such that $\bigcup_{j=1}^k B_j = [n] = \{1, \ldots, n\}$.
- Take a partition of $[0, 1]$ and generate Π randomly by taking $U_1, U_2, \ldots$ i.i.d. Uniform$[0, 1]$:



- Define $\Pi(\mathbf{X}) \equiv \sim_{\mathbf{X}}$ by

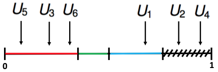
$$i \sim_{\mathbf{X}} j \quad \Longleftrightarrow \quad U_i \text{ and } U_j \text{ in same sub-interval.}$$

Theorem

$\Pi(\mathbf{X})$ *from the paintbox process is an exchangeable random partition of* $\mathbb{N}$.

Kingman's paintbox representation

Partition of $[n]$: $B_1/B_2/\cdots/B_k$ with nonempty, disjoint subsets such that $\bigcup_{j=1}^k B_j = [n]$.



Theorem (Kingman, 1978)

Let Π *be an exchangeable partition of* $\mathbb{N}$. *Then there exists a unique probability measure* ϕ *on*

$$\mathcal{F}_{\mathbb{N}}^{\downarrow} = \{(f_0, f_1, \ldots) : \sum_{i \geq 0} f_i = 1 \text{ and } f_1 \geq f_2 \geq \cdots \geq 0\}$$

so that Π *can be generated by* $\epsilon_\phi(\cdot) = \int_{\mathcal{F}_{\mathbb{N}}^{\downarrow}} \epsilon_f(\cdot)\phi(df)$:

- $f \sim \phi$,
- $X_1, X_2, \ldots$ *conditionally i.i.d. from* $P(X_i = j \mid f) = f_j$, $j \geq 0$.
- *Let* Π *be partition induced by* $(X_1, X_2, \ldots)$.

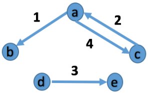
Example: $(X_1, X_2, \ldots) = (3, 0, 1, 0, 1, 1) \implies \Pi = \{1\}/\{2\}/\{3, 5, 6\}/\{4\}$

Interaction sampling

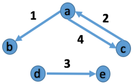
- Sample phone calls (interactions) from database:

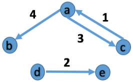
$$\underbrace{a \to b}_{X_1}, \quad \underbrace{c \to a}_{X_2}, \quad \underbrace{d \to e}_{X_3}, \quad \underbrace{a \to c}_{X_4}, \quad \cdots$$

- Represent $X_1, X_2, \ldots$ (sequence of edges) by



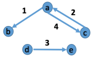
- $(X_1, X_2, \ldots) =_{\mathcal{D}} (X_{\sigma(1)}, X_{\sigma(2)}, \ldots)$ implies equal probability for



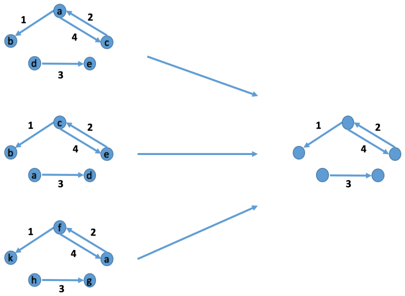


Edge-labeled structure

- Call sequence:



- Phone call network: treats only structure induced by observed calls. (Vertices identified only through their edge relations to one another.)



- Formally,

$$\mathcal{E}_{\mathbf{X}} = \{\mathbf{X}' \mid \exists \rho : \{a, b, \ldots\} \to \{a, b, \ldots\} \text{ s.t. } \rho(\mathbf{X}') = \mathbf{X}\},$$

where $\rho(\mathbf{X}') = (\rho(X_1), \rho(X_2), \ldots)$.

Edge exchangeability

- Formally, edge-labeled graph induced by $\mathbf{x} = (x_i)_{i \geq 1}$:

$$\mathcal{E}_\mathbf{x} = \{\mathbf{x}' \mid \exists \rho : \{a, b, \ldots\} \to \{a, b, \ldots\} \text{ s.t. } \rho(\mathbf{x}') = \mathbf{x}\},$$

where $\rho(\mathbf{x}') = ((\rho(x'_{11}), \rho(x'_{12})), (\rho(x'_{21}), \rho(x'_{22})), \ldots)$.

- Define the **relabeling of $\mathbf{y} = \mathcal{E}_\mathbf{x}$ by** $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$\mathbf{y}^\sigma = \{\mathbf{x}' \mid \exists \rho : \{a, b, \ldots\} \to \{a, b, \ldots\} \text{ s.t. } \rho(\mathbf{x}') = \mathbf{x}^\sigma\}.$$

Definition (Edge exchangability)

$\mathbf{Y}$ *is* **edge exchangeable** *if* $\mathbf{Y}^\sigma =_\mathcal{D} \mathbf{Y}$ *for all permutations* $\sigma : \mathbb{N} \to \mathbb{N}$.

- Edge exchangeable $\Longrightarrow$ assign equal probability to

Edge exchangeability: structural implications

Theorem (Representation theorem)

Let $\mathbf{Y}$ *be an infinite edge exchangeable random graph then*

$$\mathbf{Y} \sim \epsilon_\phi = \int_{\mathcal{F}^{\downarrow}_{\mathbb{N}\times\mathbb{N}}} \epsilon_f(\cdot)\phi(df)$$

for measure ϕ *on*

$$\mathcal{F}^{\downarrow}_{\mathbb{N}\times\mathbb{N}} := \left\{ (f_{(i,j)})_{i,j\geq -1} : f_{(i,j)} \geq 0, \sum_{i,j\geq -1} f_{(i,j)} = 1, \sum_{j\geq 0} f_{(i,j)} \geq \sum_{j\geq 0} f_{(i+1,j)}, \quad i \geq 1 \right\}$$

1. Draw $f \sim \phi$,
2. Given f, draw edges i.i.d. $\mathbb{P}\{X_k = (i,j) \mid f\} = f_{(i,j)}$.

Edge types:

- isolated interactions: $f_{(i,j)}$ for $i, j \geq 0$
- one-off interaction: $f_{(i,j)}$ for $i \leq 0$ and $j \geq 1$ or $i \geq 1$ and $j \leq 0$.
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Edge exchangeability: Illustration

Theorem (Representation theorem)

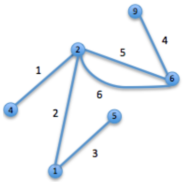
Let $\mathbf{Y}$ *be an infinite edge exchangeable random graph. Then there exists a unique probability measure* ϕ *on* $\mathcal{F}^{\downarrow}_{\mathbb{N}\times\mathbb{N}}$ *such that* $\mathbf{Y} \sim \epsilon_\phi$*, where*

$$\epsilon_\phi(\cdot) = \int_{\mathcal{F}^{\downarrow}_{\mathbb{N}\times\mathbb{N}}} \epsilon_f(\cdot)\phi(df).$$

- Generate $\mathbf{Y}$ by first sampling $f \sim \phi$ and, given $f = (f_{(i,j)})_{i,j\geq -1}$, putting $\mathbf{Y} = \mathbf{Y}(X_1, X_2, \ldots)$ for $X_1, X_2, \ldots$ i.i.d. from

$$\mathbb{P}\{X_k = (i,j) \mid f\} = f_{(i,j)}, \quad i,j \geq -1.$$

- For example: $X_1 = (2,4)$, $X_2 = (1,2)$, $X_3 = (1,5)$, $X_4 = (6,9)$, $X_5 = (2,6)$, $X_6 = (2,6)$, generates



Edge exchangeability: Open Problem

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Open Problem

Prove anything about edge exchangeability that's not already in

- *H. Crane and W. Dempsey. (2018). Edge exchangeable models for interaction networks.* Journal of the American Statistical Association.
- *H. Crane and W. Dempsey. (2019). Relational exchangeability.* Journal of Applied Probability.
- *S. Janson. (2017). On edge exchangeable graphs. arXiv:1702.06396.*

Hollywood model (C.–Dempsey, 2018)

Generate a sequence of edges $(E_i)_{i \geq 1}$ as follows.

- Label elements in order of appearance.
- $D(j)$: degree of vertex labeled j.
- Choose elements in edge i, denoted $E_i(1)$ and $E_i(2)$, by

$$\mathrm{pr}(E_i(r) = j \mid \text{past}) \propto \begin{cases} D(j) - \alpha, & j = 1, \ldots, N_i \\ \theta + \alpha N_i, & j = N_i + 1. \end{cases}$$

Any realization of n interactions $\mathbf{Y}_n = E$ occurs with probability

$$\alpha^{v(E)} \frac{(\theta/\alpha)^{\uparrow v(E)}}{\theta^{\uparrow m_n(E)}} \prod_{k=2}^{\infty} \exp\{N_k(E) \log((1-\alpha)^{\uparrow(k-1)})\},$$

where

- $v(E)$ is the number of nonisolated vertices in E,
- $(N_k(E))_{k \geq 0}$ gives the number of vertices with degree k for each $k \geq 0$,
- $M_k(E)$ is the number of k-ary edges in E,
- $m_n(E) = \sum_{k \geq 1} k M_k(E)$ is the total degree of E, and
- $x^{\uparrow j} = x(x+1) \cdots (x+j-1)$ is the ascending factorial function.

Hollywood model: Basic facts

$(\mathbf{Y}_n)_{n\geq 1}$ obeys Hollywood process with parameter (α, θ) for $0 < \alpha < 1$ and $\theta > -\alpha$.

- For each $n \geq 1$,
$$p_n(k) = N_k(\mathbf{Y}_n)/v(\mathbf{Y}_n), \quad k \geq 1$$
is the empirical degree distribution of $\mathbf{Y}_n$, where $N_k(\mathbf{Y}_n)$ is the number of vertices with degree $k \geq 1$ and $v(\mathbf{Y}_n)$ is the number of vertices in $\mathbf{Y}_n$, respectively.

Theorem (Power law (C–D, 2018))

For every $k \geq 1$,
$$p_n(k) \sim \alpha k^{-(\alpha+1)}/\Gamma(1-\alpha) \quad \text{a.s. as } n \to \infty,$$
that is, $\mathbf{Y}$ exhibits power law degree distribution with exponent $\gamma = \alpha + 1 \in (1, 2)$.

Theorem (Sparsity (C–D, 2018))

The expected number of vertices in $\mathbf{Y}_n$ satisfies
$$E(v(\mathbf{Y}_n)) \sim \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta+\alpha)}(\mu n)^{\alpha} \quad \text{as } n \to \infty, \tag{2}$$
where $\mu = \sum_{k\geq 1} k\nu_k$ is the mean edge arity. Furthermore, if $1/\mu < \alpha < 1$, then $(\mathbf{Y}_n)_{n\geq 1}$ is sparse almost surely.

$(\mathbf{Y}_n)_{n\geq 1}$ obeys Hollywood process with parameter (α, θ) for $0 < \alpha < 1$ and $\theta > -\alpha$.

- For each $n \geq 1$,

$$p_n(k) = N_k(\mathbf{Y}_n)/v(\mathbf{Y}_n), \quad k \geq 1$$

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Open Problem

Analyze other properties of graphs $(\mathbf{Y}_n)_{n\geq 1}$ generated from Hollywood(α, θ) process, e.g.,

- *distribution of triangles,*
- *distribution of component sizes,*
- *distribution/behavior of any other network statistics.*

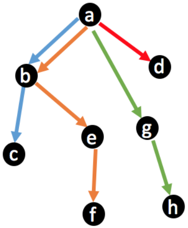
More general forms

Example: Path sampling

Sample paths between IP addresses:

$$\underbrace{a \to b \to e \to f}_{X_1}, \quad \underbrace{a \to d}_{X_2}, \quad \underbrace{a \to b \to c}_{X_3}, \quad \underbrace{a \to g \to h}_{X_4}, \quad \cdots$$

- Induced network: Relational structure observed by sampling paths representative of paths sampled by traceroute.



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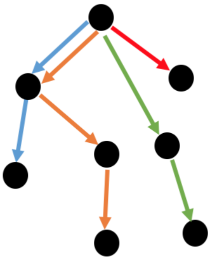

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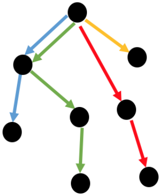
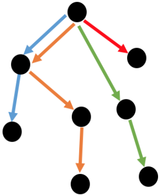
- path exchangeability: assign equal probability to





Let

- $\mathcal{R}$ be a set of relations on a finite set (singleton sets, edge, hyperedges, paths, graphs, etc.) and
- $\mathcal{F}_{\mathcal{R}}^{\downarrow}$ be the "ranked simplex" indexed by the elements of $\mathcal{R}$.

$\mathcal{R}$**-structure**: structure obtained by removing element labels (taking equivalence class) of sequence from $\mathcal{R}$. (Gluing edge, paths, etc. together.)

Theorem (C.-Dempsey, 2019)

Let Y be an infinite relationally exchangeable $\mathcal{R}$-structure. Then there exists a unique probability measure ϕ on $\mathcal{F}_{\mathcal{R}}^{\downarrow}$ such that $Y \sim \epsilon_{\phi}$, where

$$\epsilon_{\phi}(\cdot) = \int_{\mathcal{F}_{\mathcal{R}}^{\downarrow}} \epsilon_f(\cdot)\phi(df).$$

H. Crane and W. Dempsey. (2019). Relational exchangeability. Journal of Applied Probability.

Summary

Theorem (Aldous–Hoover, C.-Towsner)

Let $X = (X^1, \ldots, X^{r'})$ be relatively exchangeable with respect to $\mathfrak{M} = (\mathfrak{M}^1, \ldots, \mathfrak{M}^r)$. Then there exists $g = (g_1, \ldots, g_{r'})$ such that $\mathbf{X} =_{\mathcal{D}} \mathbf{X}^*$ with

$$X_s^{*j} = g_j(\mathfrak{M}|_{\{s_1,\ldots,s_{i'_j}\}}, (U_t)_{t\subseteq\{s_1,\ldots,s_{i'_j}\}}), \quad s = (s_1, \ldots, s_{i'_j}) \in \mathbb{N}^{i'_j}, \tag{3}$$

for $(U_t)_{t\subseteq\mathbb{N}:|t|\leq \max i'_j}$ i.i.d. Uniform$[0,1]$ and $\mathfrak{M}|_S := (\mathfrak{M}^1|_S, \ldots, \mathfrak{M}^r|_S)$.

- Aldous–Hoover: $\mathfrak{M} = \emptyset$ or any perfectly symmetric structure ($\mathrm{Aut}(\mathfrak{M}) = \mathbf{Sym}(\mathbb{N})$).
- Stochastic blockmodel: $\mathfrak{M} \in [k]^{\mathbb{N}}$ (labeled classes) or $\mathfrak{M} \in \mathcal{P}_{\mathbb{N}}$ (unlabeled).

Theorem (Kingman, C.-Dempsey)

Let Y be an infinite relationally exchangeable $\mathcal{R}$-structure. Then there exists a unique probability measure ϕ on $\mathcal{F}_{\mathcal{R}}^{\downarrow}$ such that $Y \sim \epsilon_\phi$, where

$$\epsilon_\phi(\cdot) = \int_{\mathcal{F}_{\mathcal{R}}^{\downarrow}} \epsilon_f(\cdot)\phi(df).$$

Special cases:

- Kingman: Paintbox process (random equivalence relations).
- C.-Dempsey: Edge exchangeability (random interaction networks).

References

- H. Crane and W. Dempsey. (2018). Edge exchangeable models for interaction networks. *Journal of the American Statistical Association*.
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- H. Crane and H. Towsner. (2018). Relatively exchangeable structures. *Journal of Symbolic Logic*.
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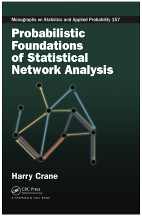
See also:

- D. Aldous. (1983). *Exchangeability and Related Topics*.



* More information available at **www.harrycrane.com/networks.html**