0. Introduction

If you had asked a probabilist in 1970 what was known about exchangeability, you would likely have received the answer "There's de Finetti's theorem: what else is there to say?" The purpose of these notes is to dispel this (still prevalent) attitude by presenting, in Parts II-IV, a variety of mostly post-1970 results relating to exchangeability. The selection of


Today: Mostly post-2015 results relating to exchangeability.
Plan

- **Lecture 1: Basic symmetries and network sampling.**

- **Lecture 2: Dynamic network models.**

Notation and Terminology

- **Network**: Abstract, non-mathematical concept. A structure of interconnected and/or interacting entities.
- **Graph**: a pair \((V, E)\) consisting of sets \(V\) (vertices) and \(E \subseteq V \times V\) (vertices).

Encoded as \(\{0, 1\}\)-valued adjacency array \(y = (y_{ij})_{i,j \in V}\) with

\[
y_{ij} = \begin{cases} 
1, & (i, j) \in E, \\
0, & \text{otherwise.}
\end{cases}
\]

- **Edge-labeled graph**: Equivalence class of structures formed out of edge sequences (formal definition later).
Variations of exchangeability

1. **Exchangeability** (conventional): $X = \mathcal{D} X^\sigma = (X_{\sigma(i)}\sigma(j))_{i,j \geq 1}$ for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$.

2. **Relative exchangeability**: $M$ represents heterogeneity of a population and $X$ is exchangeable **relative to** the symmetries of $M$.

3. **Relational exchangeability**: Exchangeability with respect to relabeling relations (species, edges, paths, networks, etc.)
Exchangeable random graphs
Vertex relabeling

- **Data:** \( Y = (Y_{ij})_{i,j \in V} \in \{0, 1\}^{V \times V}. \)
- **Symmetries:** Any permutation \( \sigma : V \to V \) determines a relabeling map
  \[
  y \mapsto y^\sigma := (Y_{\sigma(i)\sigma(j)})_{i,j \in V}.
  \]

Graph on the right obtained by relabeling graph on left with
\( \sigma(1) = 4, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 7, \sigma(5) = 3, \sigma(6) = 5, \sigma(7) = 6. \)
(Vertex) exchangeability

**Definition ((Vertex) exchangeability)**

A random graph $\mathbf{Y} = (Y_{ij})_{i,j \in V}$ is (vertex) exchangeable if

$$\mathbf{Y}^\sigma = \mathcal{D} \mathbf{Y} \quad \text{for all permutations } \sigma : V \rightarrow V.$$ 

Equivalently,

$$P(\mathbf{Y} \in A^\sigma) = P(\mathbf{Y} \in A) \quad \text{for all permutations } \sigma : V \rightarrow V,$$

where $A^\sigma = \{y^\sigma : y \in A\}$ obtained by relabeling all elements of $A$ according to $\sigma$.

- A vertex exchangeable distribution assigns equal probability to isomorphic graphs.
de Finetti’s theorem

A sequence \( X = (X_1, X_2, \ldots) \) is **exchangeable** if

\[
X^\sigma = (X_{\sigma(1)}, X_{\sigma(2)}, \ldots) = \mathcal{D} X \quad \text{for all permutations } \sigma : \mathbb{N} \to \mathbb{N}.
\]

**Theorem (de Finetti)**

Let \( X = (X_1, X_2, \ldots) \) be a countable, exchangeable \( \{0, 1\} \)-valued sequence. Then there exists a unique probability measure \( \mu \) on \([0, 1]\) such that the finite-dimensional distributions of \( X \) are given by

\[
Pr((X_1, \ldots, X_n) = (x_1, \ldots, x_n)) = \int_0^1 p^{\sum_i x_i} (1 - p)^{n-\sum_i x_i} \mu(dp).
\]

- **Intuition:**
  - Pick a coin with a random heads-probability (according to \( \mu \)).
  - Toss the coin repeatedly to generate \( X \).
- An exchangeable sequence is a mixture of i.i.d. sequences.
- Analogous theorem holds for countably exchangeable sequences in any nice enough probability space.
A sequence $\mathbf{X} = (X_1, X_2, \ldots)$ is **exchangeable** if

$$\mathbf{X}^\sigma = (X_{\sigma(1)}, X_{\sigma(2)}, \ldots) \overset{\mathcal{D}}{=} \mathbf{X} \quad \text{for all permutations } \sigma : \mathbb{N} \to \mathbb{N}.$$ 

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**Theorem (de Finetti)**

Let \( X = (X_1, X_2, \ldots) \) be an exchangeable \( \{0, 1\} \)-valued sequence. Then there exists a measurable function \( \phi : [0, 1] \times [0, 1] \to \{0, 1\} \) such that \( X \sim D X^* = (X_1^*, X_2^*, \ldots) \), where

\[
X_j^* = \phi(U_0, U_{\{j\}}), \quad j \geq 1,
\]

for \( U_0, (U_{\{i\}})_{i \geq 1} \) i.i.d. Uniform\([0, 1]\).

- Pick a coin with a random heads-probability (according to \( \mu \)).
- Toss the coin repeatedly to generate \( X \).

- Shared dependence on \( U_0 \) is the only source of dependence among variables.
- Fix \( U_0 = \alpha \), then \( X_j^* = \phi(\alpha, U_{\{j\}}) \) is i.i.d.

*Exchangeable sequence is conditionally i.i.d.*
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*Exchangeable sequence is conditionally i.i.d.*
Exchangeable random graphs

A random array \( Y = (Y_{i,j})_{i,j \geq 1} \) is **exchangeable** if

\[
Y^\sigma = (Y_{\sigma(i)\sigma(j)})_{i,j \geq 1} \equiv D Y \quad \text{for all permutations } \sigma : \mathbb{N} \to \mathbb{N}.
\]

**Theorem (Aldous–Hoover–Kallenberg)**

Let \( Y \) be the adjacency array of an exchangeable random graph with vertex set \( \mathbb{N} \). Then there exists a measurable function \( \phi : [0, 1]^4 \to \{0, 1\} \) such that

\[
Y \equiv D Y^* = (Y^*_i)_i, j \in \mathbb{N}, \text{ where }
\]

\[
Y^*_i = \phi(U_\emptyset, U_i, U_j, U_{i,j}), \quad i, j \geq 1,
\]

for \( U_\emptyset, (U_i)_i \geq 1, (U_{i,j})_{j \geq i \geq 1} \) i.i.d. Uniform[0, 1].

Decomposes structure of exchangeable random graph:
- Global effect: \( U_\emptyset \)
- Vertex effects: \( U_i, U_j \)
- Edge effects: \( U_{i,j} \)
Exchangeable random graphs

A random array \( Y = (Y_{i,j})_{i,j \geq 1} \) is exchangeable if

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Decomposes structure of exchangeable random graph:

- **Global effect**: \( U_\emptyset \) (shared by all edges)
- **Vertex effects**: \( U_{\{i\}}, U_{\{j\}} \)
- **Edge effects**: \( U_{\{i,j\}} \)
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for \( U_\emptyset, (U_{\{i\}})_i \geq 1, (U_{\{i,j\}})_j \geq i \geq 1 \) i.i.d. Uniform\([0,1]\).

Decomposes structure of exchangeable random graph:

- Global effect: \( U_\emptyset \)
- Vertex effects: \( U_{\{i\}}, U_{\{j\}} \) (shared by all edges involving given vertex)
- Edge effects: \( U_{\{i,j\}} \)
Exchangeable random graphs

A random array \( Y = (Y_{i,j})_{i,j \geq 1} \) is **exchangeable** if

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for \( U_\emptyset, (U_{\{i\}})_{i \geq 1}, (U_{\{i,j\}})_{j \geq i \geq 1} \) i.i.d. Uniform[0, 1].

---

Decomposes structure of exchangeable random graph:

- **Global effect**: \( U_\emptyset \)
- **Vertex effects**: \( U_{\{i\}}, U_{\{j\}} \)
- **Edge effects**: \( U_{\{i,j\}} \) (only for edge between \( i \) and \( j \))
Dissociated random graphs

Definition (Dissociated array)

A random array $Y = (Y_{ij})_{i,j \geq 1}$ is **dissociated** if

$Y \mid_S$ and $Y \mid_T$ are independent for all $S, T \subseteq \mathbb{N}$ such that $S \cap T = \emptyset$.

Fix $U_\emptyset = \alpha$ in Aldous–Hoover: for $S \cap T = \emptyset$:

- $Y^* \mid_S = (\phi(\alpha, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}))_{i,j \geq s}$ depends on $U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}$ indexed by $S$.
- $Y^* \mid_T = (\phi(\alpha, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}))_{i,j \geq T}$ depends on $U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}$ indexed by $T$.

*Every exchangeable random graph is a mixture of dissociated, exchangeable graphs.*

Example:

- Erdős–Rényi model: all edges are i.i.d. ($\phi$ depends only on last argument.)
- Graphon models: let $g : [0, 1] \times [0, 1] \to [0, 1]$, let $U_1, U_2, \ldots$ i.i.d. Uniform$[0, 1]$ and let

$$\mathbb{P}(Y_{ij} = 1 \mid U_i, U_j) = g(U_i, U_j), \quad i, j \geq 1.$$
Graph limits

Let $G = (G_{ij})_{i,j \in \mathbb{N}}$ be a countable graph and $F = (F_{ij})_{1 \leq i,j \leq m}$ be a graph with vertex set $[m] = \{1, \ldots, m\}$.

- For each $n \geq 1$, define
  \[
  t_n(F, G) = \frac{1}{n^m} \sum_{\text{injections } \psi:[m] \to [n]} \mathbf{1}(G^\psi = F).
  \]

- The **homomorphism density of $F$ in $G$** is the limit
  \[
  t(F, G) = \lim_{n \to \infty} t_n(F, G) \quad \text{if the limit exists.}
  \]

- $G$ possesses a **graph limit** if $t(F, G)$ exists for all finite $F$, for all $m \geq 1$.

**Corollary**

**Graph limits** $\leftrightarrow$ exchangeable, dissociated probability measures on countable graphs.

**Immediate implications:**

(i) Dense structure: Exchangeable random graph $\implies$ dense or empty w.p. 1.

(ii) Representative sampling: normalizing constant $1/n^m$ interpreted as assigning equal probability (uniform distribution) on all $\psi$-sampling maps.
Graph limits

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**Corollary**

Graph limits $\leftrightarrow$ exchangeable, dissociated probability measures on countable graphs.

**Open Problem**

*Define and study an interesting notion of asymptotics for sparse/complex networks.*
Relative exchangeability
Relative symmetries

**Definition (Relative exchangeability)**

*Invariance with respect to the symmetries of another structure.*

Population \( \mathbb{N} = \{1, 2, \ldots\} \) divides into two classes, e.g., male and female.

- Define \( C = (C_1, C_2, \ldots) \) by

\[
C_i = \begin{cases} 
1, & \text{i is male,} \\
0, & \text{otherwise.}
\end{cases}
\]

- \((X_1, X_2, \ldots)\) is relatively exchangeable with respect to \( C \), i.e., \( X^\sigma = \mathcal{D} X \) for permutations \( \sigma : \mathbb{N} \to \mathbb{N} \) that fix \( C \).

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\( X = \mathcal{D} X' \)
Relatively exchangeable sequences

Definition (Relative exchangeability)

Invariance with respect to the symmetries of another structure.

Population $\mathbb{N} = \{1, 2, \ldots\}$ divides into two classes, e.g., male and female.

- Define $C = (C_1, C_2, \ldots)$ by

  $$C_i = \begin{cases} 
  1, & \text{if } i \text{ is male,} \\
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- Measurements $(X_1, X_2, \ldots)$ are relatively exchangeable with respect to $C$, i.e., $X^{\sigma} = D X$ for permutations $\sigma : \mathbb{N} \to \mathbb{N}$ that fix $C$.

Theorem

Let $C = (C_1, C_2, \ldots)$ be $[k]$-valued sequence* and $X = (X_1, X_2, \ldots)$ be relatively exchangeable with respect to $C$. Then there exists a measurable $\phi : [k] \times [0, 1]^2 \to \{0, 1\}$ such that $X = D X^* = (X^*_i)_{i \geq 1}$ with

$$X^*_i = \phi(C_i, U_\emptyset, U_{\{i\}}), \quad i \geq 1,$$

for $U_\emptyset$ and $(U_{\{i\}})_{i \geq 1}$ i.i.d. Uniform$[0, 1]$.
Relatively exchangeable random graphs

**Definition (Relatively exchangeable random graph)**

Y is relatively exchangeable with respect to G if, for all \( S \subseteq \mathbb{N} \), \( Y|_S \sim_d Y|_S \) for all automorphisms \( \sigma \) of \( G|_S \).
Relatively exchangeable random graphs

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$Y$ is relatively exchangeable with respect to $G$ if, for all $S \subseteq \mathbb{N}$, $Y|_S \sim D Y|_S$ for all automorphisms $\sigma$ of $G|_S$.

**Theorem (C. 2017)**

Let $G = (\mathbb{N}, E)$ be an undirected graph* and $Y$ be relatively exchangeable with respect to $G$. There exists $\phi : \{0, 1\} \times [0, 1]^4 \to \{0, 1\}$ such that $Y \sim D Y^* = (Y^*_{ij})_{i,j \geq 1}$ with

$$Y^*_{ij} = \phi(G_{ij}, U_\emptyset, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad i, j \geq 1$$

where

$$G_{ij} = \begin{cases} 1, & (i, j) \in E, \\ 0, & \text{otherwise}. \end{cases}$$

- relatively exchangeable (structural) component
- exchangeable (Aldous–Hoover) component
**Definition (Relatively exchangeable random graph)**

$Y$ is relatively exchangeable with respect to $G$ if, for all $S \subseteq \mathbb{N}$, $Y |_S \overset{D}{=} Y |_S$ for all automorphisms $\sigma$ of $G|_S$.

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*relatively exchangeable (structural) component

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- relatively exchangeable (structural) component
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General case

\( \mathcal{M} \): general combinatorial structure

\( \mathcal{Y} \) is \( \mathcal{M} \)-exchangeable (exchangeable relative to \( \mathcal{M} \)):

**Sequence:** \( \mathcal{M} = (M_1, M_2, \ldots) \in \{0, 1\}^N \),

\[
Y_i = \phi(M_i, U_\emptyset, U_{\{i\}}), \quad i \geq 1.
\]

**Graph:** \( \mathcal{M} = (M_{ij})_{i,j \geq 1} \in \{0, 1\}^N \times N \),

\[
Y_{ij} = \phi(M_{ij}, U_\emptyset, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad i, j \geq 1.
\]

**Lack of interference:** Both exhibit strong local dependence on \( \mathcal{M} \).

- Does this lack of interference hold in general? No.
- What properties must \( \mathcal{M} \) satisfy to get the representation?
\\( M \): general combinatorial structure

**Y** is **M**-exchangeable (exchangeable relative to **M**):

**Sequence**: \( M = (M_1, M_2, \ldots) \in \{0, 1\}^\mathbb{N}, \)

\[
Y_i = \phi(M_i, U_\emptyset, U_{\{i\}}), \quad i \geq 1.
\]

**Graph**: \( M = (M_{ij})_{i,j \geq 1} \in \{0, 1\}^{\mathbb{N} \times \mathbb{N}}, \)

\[
Y_{ij} = \phi(M_{ij}, U_\emptyset, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad i, j \geq 1.
\]

**Lack of interference**: *Both exhibit strong local dependence on *\( M \).*

- Does lack of interference hold in general? No.
- What properties must *\( M \)* satisfy to get this representation?
Notation

General setting:
- **signature**: $\mathcal{L} = \{i_1, \ldots, i_r\}$ with $1 \leq i_1 \leq \cdots \leq i_r$.
- **$\mathcal{L}$-structure**: $\mathcal{M} = (\mathcal{M}^1, \ldots, \mathcal{M}^r)$ with each $\mathcal{M}^j$ a symmetric $i_j$-ary relation $\mathcal{M}^j \subseteq \mathbb{N}^{i_j}$.
- **adjacency array**: $\mathcal{M} = (\mathcal{M}^1, \ldots, \mathcal{M}^r)$ corresponds to a collection of $\{0, 1\}$-valued arrays $\mathcal{M}^j = (\mathcal{M}^j_s, s \in \mathbb{N}^{i_j})$ with
  \[
  \mathcal{M}^j_s = 1 \iff s \in \mathcal{M}^j.
  \]

Example:
- $\mathcal{L} = \{1, 2\}$: Graph with colored vertices

![Graph with colored vertices](image)
Notation

General setting:

- **signature**: $L = \{i_1, \ldots, i_r\}$ with $1 \leq i_1 \leq \cdots \leq i_r$.
- **$L$-structure**: $M = (M^1, \ldots, M^r)$ with each $M^i$ a symmetric $i_j$-ary relation $M^i \subseteq N_{i_j}$.
- **adjacency array**: $M = (M^1, \ldots, M^r)$ corresponds to a collection of $\{0, 1\}$-valued arrays $M^i = (M^i_s, s \in N_{i_j})$ with

\[
M^i_s = 1 \iff s \in M^i.
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Example:

- $L = \{1, 2\}$: Graph with colored vertices
Relative exchangeability

- population structure: \( M = (M^1, \ldots, M^r) \) with each \( M^i = (M^i_s, s \in \mathbb{N}^i) \) for \( i \geq 1 \).
- random structure: \( Y = (Y_{ij})_{i,j \geq 1} \).

**Definition**

**Y is relatively exchangeable with respect to** \( M \) if \( Y|_S^{\sigma} \equiv_{D} Y|_S \) for all permutations \( \sigma : S \to S \) such that \( M|_S^{\sigma} = M|_S \).

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\( M \):

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\( \{1,3,5\} \) \{2,4,6,7\}

\( X \):

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\( \{1,3\} \) \{2,6\}
Relative exchangeability

- population structure: $\mathcal{M} = (\mathcal{M}^1, \ldots, \mathcal{M}^r)$ with each $\mathcal{M}^i = (\mathcal{M}^i_s, s \in \mathbb{N}^i)$ for $i \geq 1$.
- random structure: $Y = (Y_{ij})_{i,j \geq 1}$.

**Definition**

$Y$ is relatively exchangeable with respect to $\mathcal{M}$ if $Y|_S^\sigma = D Y|_S$ for all permutations $\sigma : S \rightarrow S$ such that $\mathcal{M}|_S^\sigma = \mathcal{M}|_S$. 

![Diagram](image)
Relative exchangeability

- population structure: \( \mathcal{M} = (\mathcal{M}^1, \ldots, \mathcal{M}^r) \) with each \( \mathcal{M}^i = (\mathcal{M}^i_s, s \in \mathbb{N}^i) \) for \( i_j \geq 1 \).
- random structure: \( Y = (Y_{ij})_{i,j \geq 1} \).

**Definition**

\( Y \) is relatively exchangeable with respect to \( \mathcal{M} \) if \( Y \mid \sigma_S = \mathcal{D} Y \mid S \) for all permutations \( \sigma : S \rightarrow S \) such that \( \mathcal{M} \mid \sigma_S = \mathcal{M} \mid S \).
Let $Y = (Y_{ij})_{i,j \geq 1}$ be relatively exchangeable with respect to $\mathcal{M} = (\mathcal{M}^1, \ldots, \mathcal{M}^r)$. Then there exists $\phi$ such that $Y =_{\mathcal{D}} Y^*$ with

$$Y_{ij} = \phi(\mathcal{M}|_{\{i,j\}}, U_\emptyset, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad (i, j) \in \mathbb{N},$$

for $U_\emptyset, (U_{\{i\}})_{i \geq 1}, (U_{\{i,j\}})_{j \geq i \geq 1}$ i.i.d. Uniform[0, 1] and $\mathcal{M}|_S := (\mathcal{M}^1|_S, \ldots, \mathcal{M}^r|_S)$.

Representation in (3) holds only under strong condition on $\mathcal{M}$:

- **ultrahomogeneous**: every embedding $\mathcal{N} \rightarrow \mathcal{M}$ extends to a automorphism of $\mathcal{M}$.
- **$n$-disjoint amalgamation ($n$-DAP)**: Let $K$ (set of finite structures) be closed under isomorphism. For every $(\mathcal{G}_i)_{1 \leq i \leq n}$ satisfying
  
  - $\mathcal{G}_i \in K$,
  - $|\mathcal{G}_i| = [n] \setminus \{i\}$,
  - and $\mathcal{G}_i|_{[n]\{i,j\}} = \mathcal{G}_j|_{[n]\{i,j\}}$ for all $1 \leq i, j \leq n$,

  there exists $\mathcal{G} \in K$ with $|\mathcal{G}| = n$ such that $\mathcal{G}|_{[n]\{i\}} = \mathcal{G}_i$ for all $1 \leq i \leq n$.

- **3-DAP (sets)**: $\mathcal{G}_i \in \{\{1, 3\}, \{1, 2\}, \{2, 3\}\}$ extends to $\{1, 2, 3\}$.
- **3-DAP fails (partitions)**: $\mathcal{G}_i \in \{1/3, 12, 2/3\}$ cannot be extended to a partition of $[3]$. 

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Relational/Edge exchangeability
Species sampling

Sample animals and record their species

\[ \text{bear, deer, bear, wolf, \ldots} \]

- Element-labeled sequence: \( X_1, X_2, X_3, X_4, \ldots \)

- Relationally-labeled structure: \( \sim X \equiv \{1, 3\}, \{2\}, \{4\}, \ldots \)

Invariance:

- \((X_1, X_2, \ldots) =_D (X_{\sigma(1)}, X_{\sigma(2)}, \ldots)\): observed species representative of all species.

- \(\sim_{X \sigma} =_D \sim_X\): relation among observed species is representative of the relation of all species.
Kingman’s paintbox representation

- **Partition of** $[n]$: $\pi = B_1/B_2/\cdots/B_k$ with nonempty, disjoint subsets such that $\bigcup_{j=1}^k B_j = [n] = \{1, \ldots, n\}$.

- Take a partition of $[0, 1]$ and generate $\Pi$ randomly by taking $U_1, U_2, \ldots$ i.i.d. Uniform$[0, 1]$:

![Diagram showing the process of generating $\Pi$](image)

- Define $\Pi(X) \equiv \sim_X$ by

  $$i \sim_X j \iff U_i \text{ and } U_j \text{ in same sub-interval.}$$

**Theorem**

$\Pi(X)$ from the paintbox process is an exchangeable random partition of $\mathbb{N}$. 

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Kingman's paintbox representation

**Partition of** \([n]\): \(B_1/B_2/\cdots/B_k\) with nonempty, disjoint subsets such that \(\bigcup_{j=1}^{k} B_j = [n]\).

Theorem (Kingman, 1978)

Let \(\Pi\) be an exchangeable partition of \(\mathbb{N}\). Then there exists a unique probability measure \(\phi\) on

\[
\mathcal{F}_{\downarrow}^{\uparrow} = \{(f_0, f_1, \ldots) : \sum_{i \geq 0} f_i = 1 \text{ and } f_1 \geq f_2 \geq \cdots \geq 0\}
\]

so that \(\Pi\) can be generated by \(\epsilon_{\phi}(\cdot) = \int_{\mathcal{F}_{\downarrow}^{\uparrow}} \epsilon_f(\cdot)\phi(df)\):

- \(f \sim \phi\),
- \(X_1, X_2, \ldots \) conditionally i.i.d. from \(P(X_i = j | f) = f_j, j \geq 0\).
- Let \(\Pi\) be partition induced by \((X_1, X_2, \ldots)\).

Example: \((X_1, X_2, \ldots) = (3, 0, 1, 0, 1, 1) \implies \Pi = \{1\}/\{2\}/\{3, 5, 6\}/\{4\}\)
Interaction sampling

- Sample phone calls (interactions) from database:

\[ a \rightarrow b, \quad c \rightarrow a, \quad d \rightarrow e, \quad a \rightarrow c, \quad \ldots \]

- Represent \( X_1, X_2, \ldots \) (sequence of edges) by

\[(X_1, X_2, \ldots) = \mathcal{D}(X_{\sigma(1)}, X_{\sigma(2)}, \ldots)\] implies equal probability for
Edge-labeled structure

- Call sequence:

- Phone call network: treats only structure induced by observed calls. (Vertices identified only through their edge relations to one another.)

Formally,

\[ \mathcal{E}_X = \{ X' \mid \exists \rho : \{ a, b, \ldots \} \to \{ a, b, \ldots \} \text{ s.t. } \rho(X') = X \}, \]

where \( \rho(X') = (\rho(X_1), \rho(X_2), \ldots) \).
Formally, edge-labeled graph induced by $x = (x_i)_{i \geq 1}$:

$$E_x = \{ x' \mid \exists \rho : \{a, b, \ldots\} \rightarrow \{a, b, \ldots\} \text{ s.t. } \rho(x') = x \},$$

where $\rho(x') = ((\rho(x'_{11}), \rho(x'_{12})), (\rho(x'_{21}), \rho(x'_{22})), \ldots)$.

Define the relabeling of $y = E_x$ by $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ by

$$y^\sigma = \{ x' \mid \exists \rho : \{a, b, \ldots\} \rightarrow \{a, b, \ldots\} \text{ s.t. } \rho(x') = x^\sigma \}.$$

**Definition (Edge exchangeability)**

$Y$ is edge exchangeable if $Y^\sigma \sim_d Y$ for all permutations $\sigma : \mathbb{N} \rightarrow \mathbb{N}$.

Edge exchangeable $\implies$ assign equal probability to
Edge exchangeability: structural implications

Theorem (Representation theorem)

Let $Y$ be an infinite edge exchangeable random graph then

\[ Y \sim \epsilon_{\phi} = \int_{\mathcal{F}_{\mathbb{N} \times \mathbb{N}}^\downarrow} \epsilon_f(\cdot) \phi(df) \]

for measure $\phi$ on

\[ \mathcal{F}_{\mathbb{N} \times \mathbb{N}}^\downarrow := \left\{ (f(i,j))_{i,j \geq -1} : f(i,j) \geq 0, \sum_{i,j \geq -1} f(i,j) = 1, \sum_{j \geq 0} f(i,j) \geq \sum_{j \geq 0} f(i+1,j), \quad i \geq 1 \right\} \]

1. Draw $f \sim \phi$,
2. Given $f$, draw edges i.i.d. $\mathbb{P}\{X_k = (i,j) \mid f\} = f(i,j)$.

Edge types:

- isolated interactions: $f(i,j)$ for $i,j \geq 0$
- one-off interaction: $f(i,j)$ for $i \leq 0$ and $j \geq 1$ or $i \geq 1$ and $j \leq 0$.
- recurring interactions: $f(i,j)$ for $i,j \geq 1$. 
Theorem (Representation theorem)

Let $Y$ be an infinite edge exchangeable random graph then

$$Y \sim \epsilon_{\phi} = \int_{\mathcal{F}^{-}_\mathbb{N} \times \mathbb{N}} \epsilon_f(\cdot)\phi(df)$$

for measure $\phi$ on

$$\mathcal{F}^{-}_\mathbb{N} \times \mathbb{N} := \left\{ \left( f(i,j) \right)_{i,j \geq -1} : f(i,j) \geq 0, \sum_{i,j \geq -1} f(i,j) = 1, \sum_{j \geq 0} f(i,j) \geq \sum_{j \geq 0} f(i+1,j), \quad i \geq 1 \right\}$$

1. Draw $f \sim \phi$,
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Theorem (Representation theorem)

Let $Y$ be an infinite edge exchangeable random graph then

$$Y \sim \epsilon \phi = \int_{\mathcal{F}_N \times N} \epsilon_f(\cdot) \phi(df)$$

for measure $\phi$ on

$$\mathcal{F}_N \times N := \left\{ (f(i,j))_{i,j \geq -1} : f(i,j) \geq 0, \sum_{i,j \geq -1} f(i,j) = 1, \sum_{j \geq 0} f(i,j) \geq \sum_{j \geq 0} f(i+1,j), \ i \geq 1 \right\}$$

1. Draw $f \sim \phi$,
2. Given $f$, draw edges i.i.d. $\mathbb{P}\{X_k = (i,j) \mid f\} = f(i,j)$.

Edge types:
- isolated interactions: $f(i,j)$ for $i, j \geq 0$
- one-off interaction: $f(i,j)$ for $i \leq 0$ and $j \geq 1$ or $i \geq 1$ and $j \leq 0$.
- recurring interactions: $f(i,j)$ for $i, j \geq 1$. 

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Theorem (Representation theorem)

Let $Y$ be an infinite edge exchangeable random graph. Then there exists a unique probability measure $\phi$ on $\mathcal{F}_\mathbb{N} \times \mathbb{N}$ such that $Y \sim \epsilon_{\phi}$, where

$$
\epsilon_{\phi}(\cdot) = \int_{\mathcal{F}_\mathbb{N} \times \mathbb{N}} \epsilon_f(\cdot) \phi(df).
$$

- Generate $Y$ by first sampling $f \sim \phi$ and, given $f = (f(i,j))_{i,j \geq -1}$, putting $Y = Y(X_1, X_2, \ldots)$ for $X_1, X_2, \ldots$ i.i.d. from

$$
\mathbb{P}\{X_k = (i, j) \mid f\} = f(i,j), \quad i, j \geq -1.
$$

- For example: $X_1 = (2, 4), X_2 = (1, 2), X_3 = (1, 5), X_4 = (6, 9), X_5 = (2, 6), X_6 = (2, 6)$, generates

![Graph Diagram]
Theorem (Representation theorem)

Let $Y$ be an infinite edge exchangeable random graph. Then there exists a unique probability measure $\phi$ on $\mathcal{F}_\mathbb{N}^\perp \times \mathbb{N}$ such that $Y \sim \epsilon_\phi$, where

$$
\epsilon_\phi(\cdot) = \int_{\mathcal{F}_\mathbb{N}^\perp \times \mathbb{N}} \epsilon_f(\cdot) \phi(df).
$$

Open Problem

Prove anything about edge exchangeability that’s not already in


Generate a sequence of edges \((E_i)_{i \geq 1}\) as follows.

- Label elements in order of appearance.
- \(D(j)\): degree of vertex labeled \(j\).
- Choose elements in edge \(i\), denoted \(E_i(1)\) and \(E_i(2)\), by

\[
\text{pr}(E_i(r) = j \mid \text{past}) \propto \begin{cases} 
D(j) - \alpha, & j = 1, \ldots, N_i \\
\theta + \alpha N_i, & j = N_i + 1.
\end{cases}
\]

Any realization of \(n\) interactions \(Y_n = E\) occurs with probability

\[
\alpha^{\nu(E)} \left(\frac{\theta}{\alpha}\right)^{\nu(E)} \prod_{k=2}^{\infty} \exp\{N_k(E) \log((1 - \alpha)^{(k-1)})\},
\]

where

- \(\nu(E)\) is the number of nonisolated vertices in \(E\),
- \((N_k(E))_{k \geq 0}\) gives the number of vertices with degree \(k\) for each \(k \geq 0\),
- \(M_k(E)\) is the number of \(k\)-ary edges in \(E\),
- \(m_n(E) = \sum_{k \geq 1} kM_k(E)\) is the total degree of \(E\), and
- \(x^{\uparrow j} = x(x + 1) \cdots (x + j - 1)\) is the ascending factorial function.
Hollywood model: Basic facts

\((Y_n)_{n \geq 1}\) obeys Hollywood process with parameter \((\alpha, \theta)\) for \(0 < \alpha < 1\) and \(\theta > -\alpha\).

- For each \(n \geq 1\),
  \[
p_n(k) = \frac{N_k(Y_n)}{v(Y_n)}, \quad k \geq 1
  \]
  is the empirical degree distribution of \(Y_n\), where \(N_k(Y_n)\) is the number of vertices with degree \(k \geq 1\) and \(v(Y_n)\) is the number of vertices in \(Y_n\), respectively.

**Theorem (Power law (C–D, 2018))**

For every \(k \geq 1\),
\[
p_n(k) \sim \alpha k^{-(\alpha+1)}/\Gamma(1-\alpha) \quad \text{a.s. as } n \to \infty,
\]
that is, \(Y\) exhibits power law degree distribution with exponent \(\gamma = \alpha + 1 \in (1, 2)\).

**Theorem (Sparsity (C–D, 2018))**

The expected number of vertices in \(Y_n\) satisfies
\[
E(v(Y_n)) \sim \frac{\Gamma(\theta + 1)}{\alpha \Gamma(\theta + \alpha)} (\mu n)^\alpha \quad \text{as } n \to \infty,
\]
where \(\mu = \sum_{k \geq 1} k \nu_k\) is the mean edge arity. Furthermore, if \(1/\mu < \alpha < 1\), then \((Y_n)_{n \geq 1}\) is sparse almost surely.
For each $n \geq 1$, 

$$p_n(k) = \frac{N_k(Y_n)}{v(Y_n)}, \quad k \geq 1$$

is the empirical degree distribution of $Y_n$, where $N_k(Y_n)$ is the number of vertices with degree $k \geq 1$ and $v(Y_n)$ is the number of vertices in $Y_n$, respectively.

Open Problem

Analyze other properties of graphs $(Y_n)_{n \geq 1}$ generated from Hollywood($\alpha, \theta$) process, e.g.,

- distribution of triangles,
- distribution of component sizes,
- distribution/behavior of any other network statistics.
More general forms
Example: Path sampling

Sample paths between IP addresses:

\[ a \rightarrow b \rightarrow e \rightarrow f, \quad a \rightarrow d, \quad a \rightarrow b \rightarrow c, \quad a \rightarrow g \rightarrow h, \quad \cdots \]

- Induced network: Relational structure observed by sampling paths representative of paths sampled by traceroute.
Example: Path sampling

Sample paths between IP addresses:

\[ a \rightarrow b \rightarrow e \rightarrow f, \quad a \rightarrow d, \quad a \rightarrow b \rightarrow c, \quad a \rightarrow g \rightarrow h, \quad \cdots \]

- Induced network: Relational structure observed by sampling paths representative of paths sampled by traceroute.
Example: Path sampling

Sample paths between IP addresses:

\[ \begin{align*}
& a \rightarrow b \rightarrow e \rightarrow f, \\
& a \rightarrow d, \\
& a \rightarrow b \rightarrow c, \\
& a \rightarrow g \rightarrow h, \\
& \cdots
\end{align*} \]

- Induced network: Relational structure observed by sampling paths representative of paths sampled by traceroute.
- path exchangeability: assign equal probability to
Relational Exchangeability

Let

- $\mathcal{R}$ be a set of relations on a finite set (singleton sets, edge, hyperedges, paths, graphs, etc.) and
- $\mathcal{F}_\mathcal{R}^\perp$ be the “ranked simplex” indexed by the elements of $\mathcal{R}$.

$\mathcal{R}$-structure: structure obtained by removing element labels (taking equivalence class) of sequence from $\mathcal{R}$. (Gluing edge, paths, etc. together.)

Theorem (C.-Dempsey, 2019)

Let $Y$ be an infinite relationally exchangeable $\mathcal{R}$-structure. Then there exists a unique probability measure $\phi$ on $\mathcal{F}_\mathcal{R}^\perp$ such that $Y \sim \epsilon_\phi$, where

$$\epsilon_\phi(\cdot) = \int_{\mathcal{F}_\mathcal{R}^\perp} \epsilon_f(\cdot) \phi(df).$$

Theorem (Aldous–Hoover, C.-Towsner)

Let $X = (X^1, \ldots, X^{r'})$ be relatively exchangeable with respect to $\mathcal{M} = (\mathcal{M}^1, \ldots, \mathcal{M}^r)$. Then there exists $g = (g_1, \ldots, g_{r'})$ such that $X = \mathcal{D} X^*$ with

$$X^*_s^j = g_j(\mathcal{M}|_{\{s_1, \ldots, s^j_s\}}, (U_t)_{t \subseteq \{s_1, \ldots, s^j_s\}}), \quad s = (s_1, \ldots, s^j_s) \in \mathbb{N}^{i_j'}, \quad \text{(3)}$$

for $(U_t)_{t \subseteq \mathbb{N}: |t| \leq \max_i i_j'}$ i.i.d. Uniform$[0, 1]$ and $\mathcal{M}|_s := (\mathcal{M}^1|_s, \ldots, \mathcal{M}^r|_s)$.

- **Aldous–Hoover**: $\mathcal{M} = \emptyset$ or any perfectly symmetric structure ($\text{Aut}(\mathcal{M}) = \text{Sym}(\mathbb{N})$).
- **Stochastic blockmodel**: $\mathcal{M} \in [k]^{\mathbb{N}}$ (labeled classes) or $\mathcal{M} \in \mathcal{P}_{\mathbb{N}}$ (unlabeled).

Theorem (Kingman, C.-Dempsey)

Let $Y$ be an infinite relationally exchangeable $\mathcal{R}$-structure. Then there exists a unique probability measure $\phi$ on $\mathcal{F}_{\mathcal{R}}$ such that $Y \sim \epsilon_\phi$, where

$$\epsilon_\phi(\cdot) = \int_{\mathcal{F}_{\mathcal{R}}} \epsilon_f(\cdot) \phi(df).$$

**Special cases:**
- **Kingman**: Paintbox process (random equivalence relations).
- **C.-Dempsey**: Edge exchangeability (random interaction networks).
References


See also: