Probabilistic Symmetry and Network Models
Lecture 2

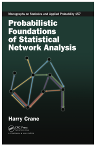
Harry Crane

October 10, 2019

Plan

- Lecture 1: Basic symmetries and network sampling.
  - H. Crane and W. Dempsey. (2018). Edge exchangeable models for interaction networks. *Journal of the American Statistical Association*.
  - H. Crane and W. Dempsey. (2019). Relational exchangeability. *Journal of Applied Probability*.
  - H. Crane and H. Towsner. (2018). Relatively exchangeable structures. *Journal of Symbolic Logic*.

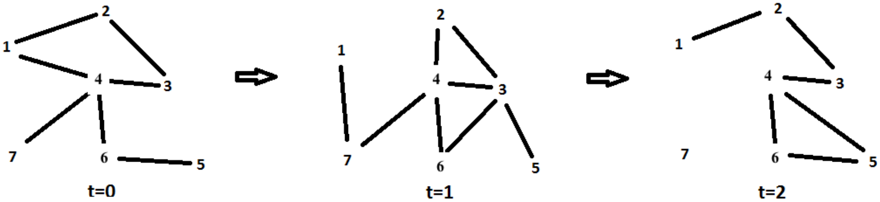
- **Lecture 2: Dynamic network models.**
  - H. Crane. (2018). *Probabilistic Foundations of Statistical Network Analysis*.
  - H. Crane. (2015). Time-varying network models. *Bernoulli*, **21**(3):1670–1696.
  - H. Crane. (2016). Dynamic random networks and their graph limits. *Annals of Applied Probability*.
  - H. Crane. (2017). Exchangeable graph-valued Feller processes. *Probability Theory and Related Fields*.
  - H. Crane. (2018). Combinatorial Lévy processes. *Annals of Applied Probability*.
  - H. Crane and H. Towsner. (2019+). The structure of combinatorial Markov processes.



Book website: `http://www.harrycrane.com/networks.html`

Time-varying network models

- $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$: graphs with vertex set $\mathbb{N} = \{1,2,\ldots\}$.
- $\{0,1\}^{n\times n}$: graphs with vertex set $[n] := \{1,\ldots,n\}$.



Basic assumptions:

$(\Gamma_t)_{t\geq 0}$ *is a Markov process on* $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ *satisfying*

- exchangeability*: for all* $\sigma : \mathbb{N}\to\mathbb{N}$*, the transition probabilities satisfy*

$$\mathbb{P}\{\Gamma_{t+1}\in\cdot \mid \Gamma_t = G\} = \mathbb{P}\{\Gamma^{\sigma}_{t+1}\in\cdot \mid \Gamma^{\sigma}_t = G\}, \quad \textit{for all } G\in\{0,1\}^{\mathbb{N}\times\mathbb{N}}.$$

- projective Markov property*:* $(\Gamma_t|_{[n]})_{t\geq 0}$ *is a Markov chain on* $\{0,1\}^{n\times n}$*, for every* $n = 1,2,\ldots$.

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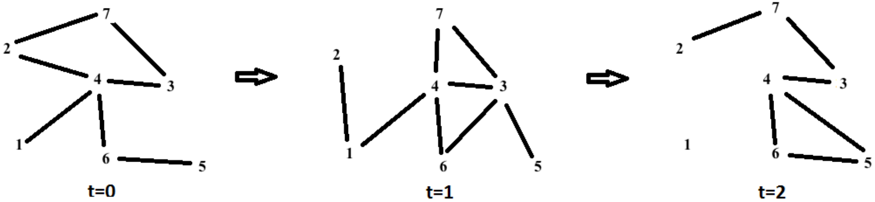

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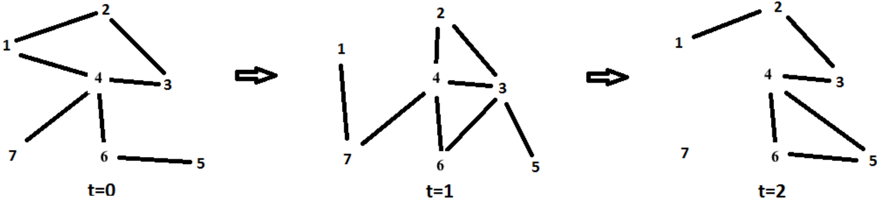

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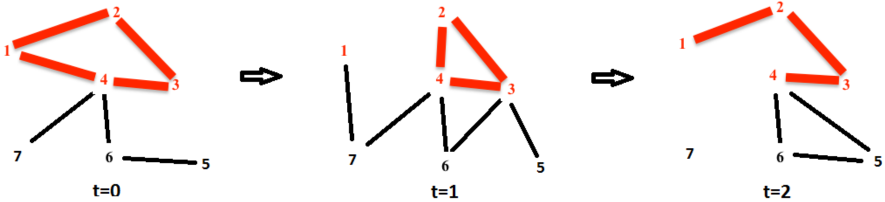

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- ***projective Markov property***: $(\Gamma_t|_{[n]})_{t\geq 0}$ **is a Markov chain on** $\{0,1\}^{n\times n}$**, for every** $n = 1, 2, \ldots$**.**

Exchangeable, Feller $\Longleftrightarrow$ Exchangeable, projective

The **semigroup** $(\mathbf{P}_t)_{t \in T}$ of a Markov process Γ acts on bounded measurable functions $g : \{0,1\}^{\mathbb{N}\times\mathbb{N}} \to \mathbb{R}$ by

$$\mathbf{P}_t g(G) := \mathbb{E}(g(\Gamma_t) \mid \Gamma_0 = G), \quad G \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}.$$

Definition (Feller property)

We say that Γ *possesses the* **Feller property** *if for all bounded, continuous* $g : \{0,1\}^{\mathbb{N}\times\mathbb{N}} \to \mathbb{R}$,

(i) $G \mapsto \mathbf{P}_t g(G)$ *is continuous for every* $t > 0$ *and*

(ii) $\lim_{t \downarrow 0} \mathbf{P}_t g(G) = g(G)$ *for all* $G \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$.

Theorem

The following are equivalent for any exchangeable Markov process Γ *on* $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$.

(i) Γ *has the projective Markov property.*

(ii) Γ *has the Feller property.*

Proof:

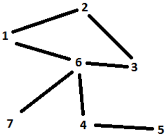
(i) $\Rightarrow$ (ii): Compactness + Stone–Weierstrass theorem.

(ii) $\Rightarrow$ (i): Use definition of Feller on test functions $\psi_F(G) := \mathbf{1}(G|_{[n]} = F\}$.

Rewiring operator

A $\{0,1\} \times \{0,1\}$-valued array $W = (W_{ij})_{i,j \geq 1}$ defines a map $\{0,1\}^{\mathbb{N} \times \mathbb{N}} \to \{0,1\}^{\mathbb{N} \times \mathbb{N}}$, $G \mapsto W(G) = (G'_{ij})_{i,j \geq 1}$ with

$$G'_{ij} = \begin{cases} W_{ij}(0), & G_{ij} = 0, \\ W_{ij}(1), & G_{ij} = 1. \end{cases}$$



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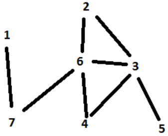
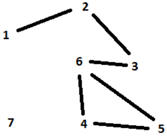


W_0



W_1

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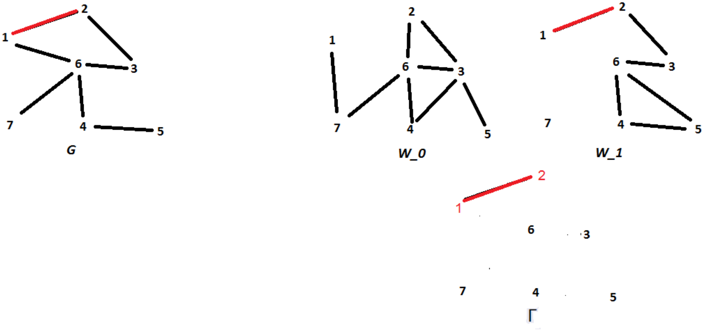

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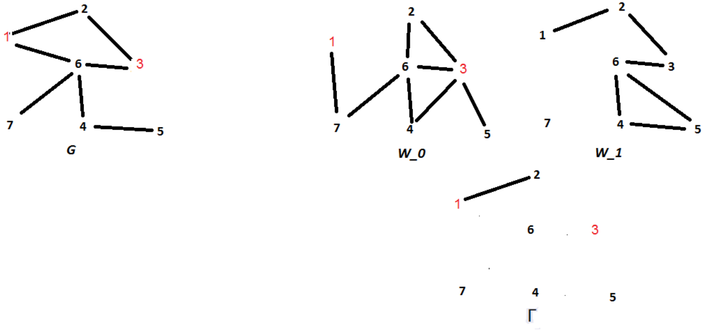

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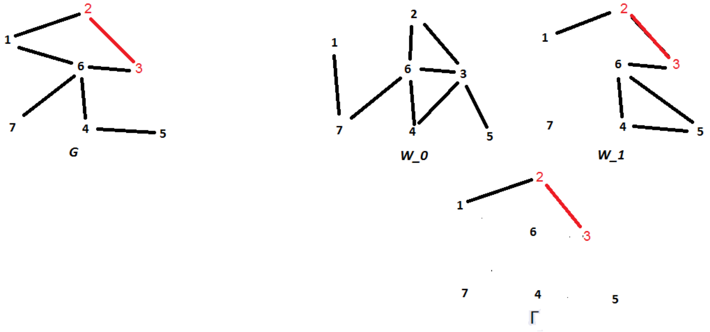

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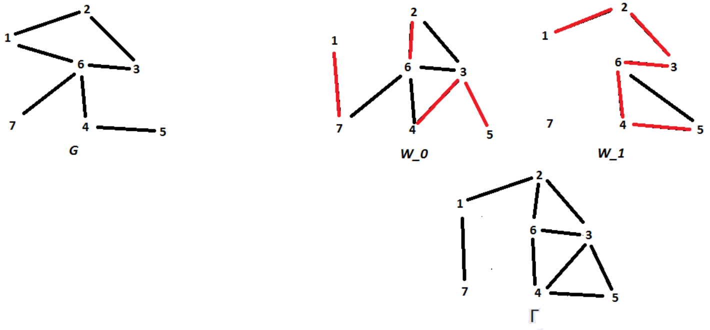

- $G \mapsto W(G)$ is Lipschitz continuous.

Exchangeable rewiring chain

Random W defines a transition probability on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by

$$P(G,\cdot) := \mathbb{P}(\{W : W(G) \in \cdot\}), \quad G \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}.$$

Let ω be an exchangeable probability measure on rewiring arrays.

For initial state $G \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, construct $\Gamma_G = (\Gamma_m)_{m\geq 0}$ by

1. taking $W_1, W_2, \ldots$ i.i.d. ω and, given $W_1, W_2, \ldots$, putting
2. $\Gamma_0 = G$ and
3. $\Gamma_{m+1} = W_{m+1}(\Gamma_m)$ for $m \geq 0$.

Define the ω**-rewiring chain with initial distribution** ν by $\Gamma^{\omega,\nu} = \Gamma_G$, for G chosen from initial distribution ν.

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Proposition

For ν an exchangeable distribution on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$, $\Gamma^{\omega,\nu}$ is exchangeable and has the Feller property.

Proof.

Exercise or see H. Crane. (2015). Time-varying network models. *Bernoulli.* □

Key claim: the converse holds.

Characterization of graph-valued Markov processes

Let $(\Gamma_m)_{m=0,1,\ldots}$ be a discrete-time Markov chain on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ (countable graphs) which is

- **exchangeable**: $(\Gamma_m)_{m\geq 0} =_{\mathcal{D}} (\Gamma_m^{\sigma})_{m\geq 0}$ for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$.
- **projective**: $(\Gamma_m|_{[n]})_{m\geq 0}$ is a Markov chain for all $n \geq 1$.

Theorem (Crane 2017)

Then there exists an exchangeable probability measure ω on $\{0,1\} \times \{0,1\}$-valued arrays so that $(\Gamma_m) =_{\mathcal{D}} (\Gamma_m^*)$ with $\Gamma_0^* =_{\mathcal{D}} \Gamma_0$ and for each $m \geq 1$

$$\Gamma_m^* = W_m(\Gamma_{m-1}^*) = (W_m \circ \cdots \circ W_1)(\Gamma_0^*), \quad m \geq 1,$$

for $W_1, W_2, \ldots$ i.i.d. ω and $G' = W(G)$ defined by

$$G'_{ij} = \begin{cases} W_{ij}^0, & G_{ij} = 0, \\ W_{ij}^1, & G_{ij} = 1. \end{cases}$$

Proof.

Exchangeable and Feller $\Rightarrow$ Γ_{t+1} is relatively exchangeable with respect to Γ_t for each t. Use representation for relatively exchangeable graphs to construct ω. □

Recall: Exchangeable random graphs (Aldous, 1979; Hoover, 1981)

$\Gamma = (\Gamma_{ij})_{i,j\geq 1}$: adjacency array of a random graph with vertex set $\mathbb{N}$.

Definition (Exchangeable random graph)

Γ *is* exchangeable *if* $\Gamma^{\sigma} = (\Gamma_{\sigma(i)\sigma(j)})_{i,j\geq 1} =_{\mathcal{D}} \Gamma$ *for all permutations* $\sigma : \mathbb{N} \to \mathbb{N}$.

Theorem (Aldous–Hoover)

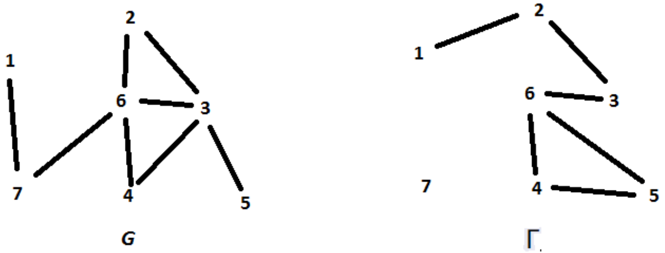
There exists a measurable function $\phi : [0,1]^4 \to \{0,1\}$ *such that* $\Gamma =_{\mathcal{D}} \Gamma^* = (\Gamma^*_{ij})_{i,j\geq 1}$ *with*

$$\Gamma^*_{ij} = \phi(U_{\emptyset}, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad i,j \geq 1,$$

for $U_{\emptyset}, (U_{\{i\}})_{i\geq 1}, (U_{\{i,j\}})_{j\geq i\geq 1}$ *i.i.d. Uniform*$[0,1]$.

Definition (Relatively exchangeable random graph)

Γ *is* relatively exchangeable with respect to *G if, for all* $S \subseteq \mathbb{N}$, $\Gamma|_S^\sigma =_{\mathcal{D}} \Gamma|_S$ *for all automorphisms* σ *of* $G|_S$.



Relative exchangeability (Crane (2017), C.–Towsner (2018))

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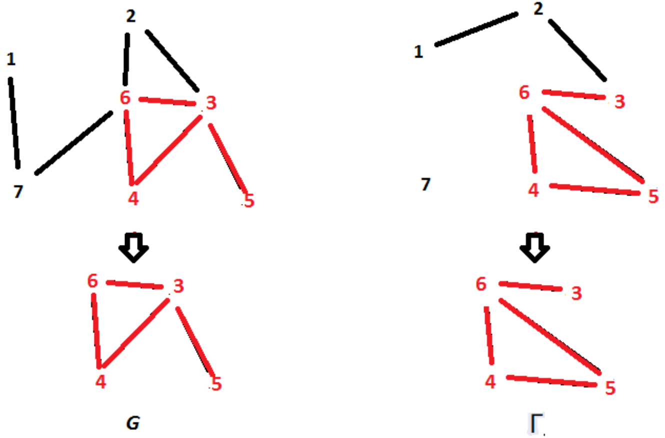

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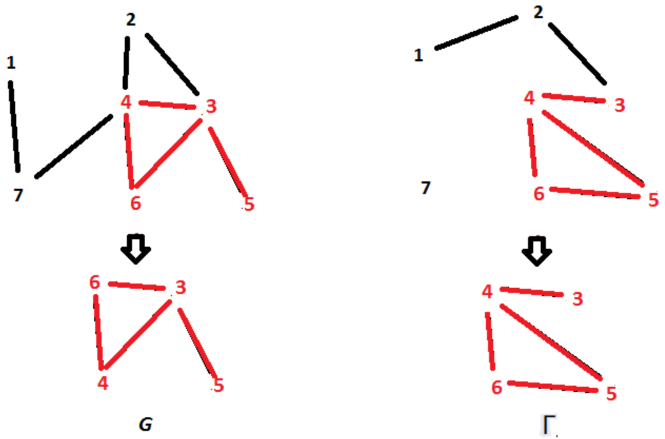
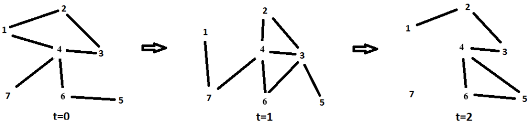


Exchangeable + Feller ⇔ Relatively exchangeable transition probability



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- exchangeability*: for all* $\sigma : \mathbb{N}\to\mathbb{N}$, $(\Gamma_t^{\sigma})_{t\geq 0}$ *has same finite-dimensional distributions as* Γ*.*
- projective Markov property*:* $(\Gamma_t|_{[n]})_{t\geq 0}$ *is a Markov chain on* $\{0,1\}^{n\times n}$*, for every* $n = 1, 2, \ldots$.

Transition from G **to** Γ:

Observation

Exchangeability and projectivity $\Longrightarrow$ *for all* $S\subseteq\mathbb{N}$*, the conditional distribution of* $\Gamma|_S$ *given* G *is invariant under automorphisms of* $G|_S$*.*

Application of representation theorem

Theorem (Crane (2017))

$G = (\mathbb{N}; E)$ an undirected graph* and Γ a random graph relatively exchangeable with respect to G. Then there exists $\phi : \{0,1\} \times [0,1]^4 \to \{0,1\}$ such that $\Gamma =_{\mathcal{D}} \Gamma^* = (\Gamma^*_{ij})_{i,j\geq 1}$ with

$$\Gamma^*_{ij} = \phi(G_{ij}, U_\emptyset, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}}), \quad i,j \geq 1,$$

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Application:

- Generate a pair of graphs $W = (W_0, W_1)$ (jointly exchangeable) by

$$W_0(i,j) = \phi(0, \xi_\emptyset, \xi_{\{i\}}, \xi_{\{j\}}, \xi_{\{i,j\}}) \quad \text{and}$$
$$W_1(i,j) = \phi(1, \xi_\emptyset, \xi_{\{i\}}, \xi_{\{j\}}, \xi_{\{i,j\}}).$$

- $W = (W_0, W_1)$ defines a random operator $\{0,1\}^{\mathbb{N}\times\mathbb{N}} \to \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by $G \mapsto G' = W(G)$ with

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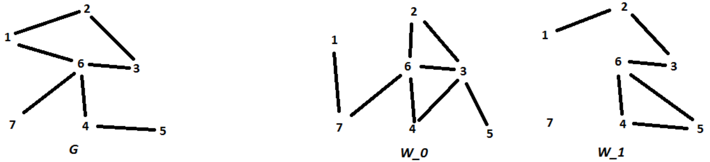

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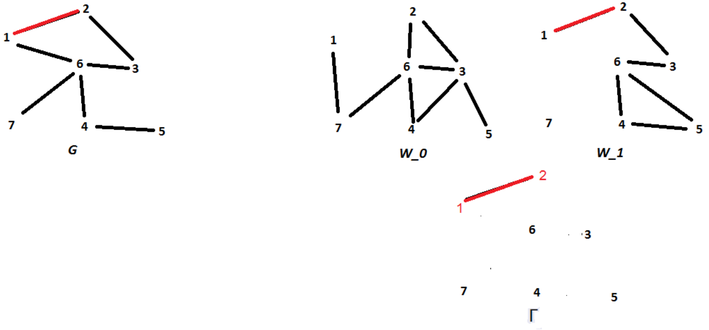

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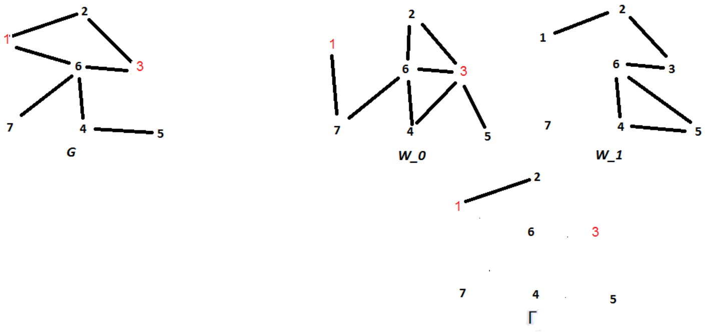

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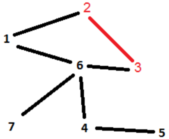

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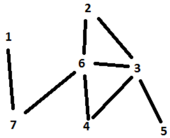
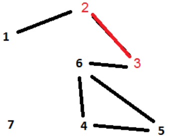
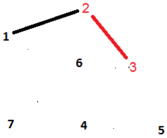


W_0



W_1



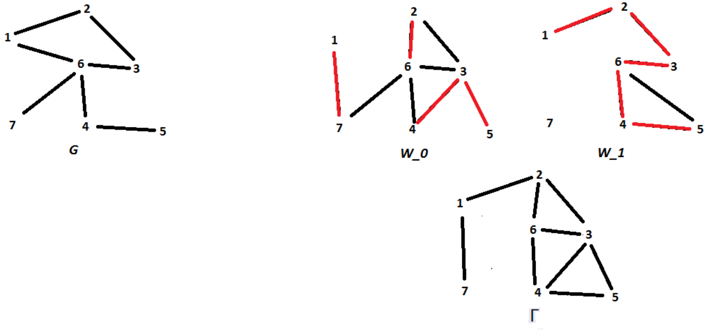
Γ

Theorem (Crane (2017))

$G = (\mathbb{N}; E)$ an undirected graph* and Γ a random graph relatively exchangeable with respect to G. Then there exists $\phi : \{0,1\} \times [0,1]^4 \to \{0,1\}$ such that $\Gamma =_{\mathcal{D}} \Gamma^* = (\Gamma^*_{ij})_{i,j\geq 1}$ with $\Gamma^*_{ij} = \phi(G_{ij}, U_\emptyset, U_{\{i\}}, U_{\{j\}}, U_{\{i,j\}})$ for $i,j \geq 1$.

Application:

- Generate a pair of graphs (W_0, W_1) (jointly exchangeable).



Sketch of proof

- Let $\Gamma = (\Gamma(t))_{t=0,1,\ldots}$ be exchangeable and have the projective Markov property.
- Draw G from Erdős–Rényi(1/2) and consider $\mathbb{P}\{\Gamma(t+1) \in \cdot \mid \Gamma(t) = G\}$.
- Given $\Gamma(t) = G$, $\Gamma(t+1)$ is relatively exchangeable with respect to G.
  $\Longrightarrow \phi : \{0,1\} \times [0,1]^4 \to \{0,1\}$ such that

$$\Gamma_{ij}(t+1) =_{\mathcal{D}} (\phi(G_{ij}, \xi_{\emptyset}, \xi_{\{i\}}, \xi_{\{j\}}, \xi_{\{i,j\}}))_{i,j\geq 1}.$$

- Now construct $W = (W_{ij})_{i,j\geq 1}$ with $W_{ij} = (W_{ij}(0), W_{ij}(1))$ by

$$W_{ij}(0) = \phi(0, \xi_{\emptyset}, \xi_{\{i\}}, \xi_{\{j\}}, \xi_{\{i,j\}})$$
$$W_{ij}(1) = \phi(1, \xi_{\emptyset}, \xi_{\{i\}}, \xi_{\{j\}}, \xi_{\{i,j\}}).$$

  Let ω be the induced probability law of W.
- For any $G' \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, there exists (w.p. 1) $\psi_{G',G} : \mathbb{N} \to \mathbb{N}$ such that $G^{\psi_{G',G}} = G'$.
- Given $W \sim \omega$ and any $G' \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, we have

$$\begin{aligned} \mathbb{P}\{W(G') \in \cdot\} &= \mathbb{P}\{\Gamma^{\psi_{G',G}}(t+1) \in \cdot \mid \Gamma(t) = G\} \\ &= \mathbb{P}\{\Gamma^{\psi_{G',G}}(t+1) \in \cdot \mid \Gamma^{\psi_{G',G}}(t) = G'\} = P(G', \cdot) \end{aligned}$$

  by the exchangeability & Feller property.

- Let ω be an exchangeable σ**-finite** measure on $\mathcal{W}_{\mathbb{N}}$, i.e.,

$$\omega(\{\mathbf{Id}_{\mathbb{N}}\}) = \mathbf{0} \quad \text{and} \quad \omega(\{\mathbf{W} \in \mathcal{W}_{\mathbb{N}} : \mathbf{W}|_{[2]} \neq \mathbf{Id}_{[2]}\}) < \infty.$$

- Let $\mathbf{W} := \{(t, W_t)\} \subseteq (0, \infty) \times (\{0,1\} \times \{0,1\})^{\mathbb{N}}$ be a Poisson point process with intensity $dt \otimes \omega$.
- Given $\mathbf{W}$, construct $\Gamma^*_{\omega} = (\Gamma^*(t))_{t \geq 0}$ with initial state $G \in \{0,1\}^{\mathbb{N} \times \mathbb{N}}$ by putting $\Gamma^*(0) = G$ and for each $n \geq 1$ and $t > 0$,
  - $\Gamma^{*[n]}_t = W^{[n]}_t(\Gamma^{*[n]}_{t-})$, if $t > 0$ is an atom of $\mathbf{W}$ for which $W^{[n]}_t \neq \mathbf{Id}_{[n]}$, or
  - $\Gamma^{*[n]}_t = \Gamma^{*[n]}_{t-}$, otherwise.

Theorem

Γ^*_{ω} *constructed above is exchangeable and Feller.*

Key claim: Converse holds.

Theorem (Crane 2017)

$\Gamma = (\Gamma_t)_{t\geq 0}$ an exchangeable, projective Markov process on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Then there exists an exchangeable measure ω satisfying

$$\omega(\{\mathbf{Id}_{\mathbb{N}}\}) = 0 \quad \textit{and} \quad \omega(\{W \in \mathcal{W}_{\mathbb{N}} : W|_{[2]} \neq \mathbf{Id}_{[2]}\}) < \infty$$

such that $\Gamma =_{\mathcal{D}} \Gamma^*_\omega$ as constructed above.

Conditions on ω:

- (identifiability) $\omega(\{\mathbf{Id}_{\mathbb{N}}\}) = 0$: $\mathbf{Id}_{\mathbb{N}}(G) = G$ for all G.
- (σ-finite) $\omega(\{W \in \mathcal{W}_{\mathbb{N}} : W|_{[2]} \neq \mathbf{Id}_{[2]}\}) < \infty$: ensures

$$\begin{aligned}
\omega(\{W|_{[n]} \neq \mathbf{Id}_{[n]}\}) &= \omega\left(\bigcup_{1\leq i,j\leq n} \{W|_{\{i,j\}} \neq \mathbf{Id}_{\{i,j\}}\}\right) \\
&\leq \sum_{1\leq i,j\leq n} \omega(\{W|_{\{i,j\}} \neq \mathbf{Id}_{\{i,j\}})\\
&= n^2\omega(W|_{[2]} \neq \mathbf{Id}_{[2]}) < \infty.
\end{aligned}$$

Lévy–Itô–Khintchine representation

Theorem (C, 2017)

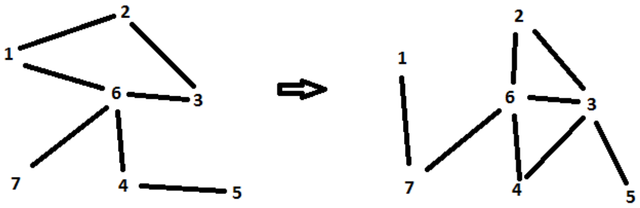
The measure ω above can be characterized by unique constants $\mathbf{e}_0, \mathbf{e}_1, \mathbf{v} \geq 0$, a unique probability measure Σ on 2×2 stochastic matrices, and a unique measure Υ on the space of graph limits satisfying

$$\Upsilon(\{\mathbf{I}\}) = 0 \quad \textit{and} \quad \int (1 - \upsilon_*^{(2)})\Upsilon(d\upsilon) < \infty,$$

such that

$$\omega = \underbrace{\Omega_\Upsilon}_{\mu_I} + \mathbf{v}\Omega_\Sigma + \mathbf{e}_0\epsilon_0 + \mathbf{e}_1\epsilon_1.$$

(I) **global jump**: a positive fraction of all edges change



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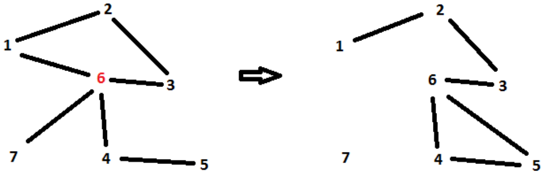
(II) **single-vertex jump**: a positive fraction of edges incident to a single vertex change, everything else stays the same



Theorem (C, 2017)

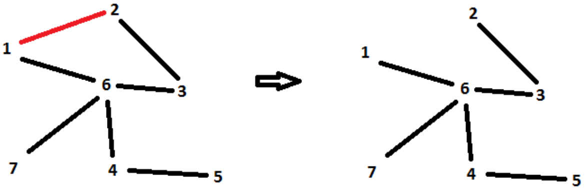
The measure ω *above can be characterized by unique constants* $\mathbf{e}_0, \mathbf{e}_1, \mathbf{v} \geq 0$*, a unique probability measure* Σ *on* 2×2 *stochastic matrices, and a unique measure* Υ *on the space of graph limits such that*

$$\Upsilon(\{\mathbf{I}\}) = 0 \quad \textit{and} \quad \int (1 - v_*^{(2)})\Upsilon(dv) < \infty,$$

such that

$$\omega = \underbrace{\Omega_\Upsilon}_{\mu_I} + \underbrace{\mathbf{v}\Omega_\Sigma}_{\mu_{II}} + \underbrace{\mathbf{e}_0\epsilon_0 + \mathbf{e}_1\epsilon_1}_{\mu_{III}}.$$

(III) **single-edge flip**: a single edge changes, everything else stays the same.



Continuous-time processes (non-Feller version)

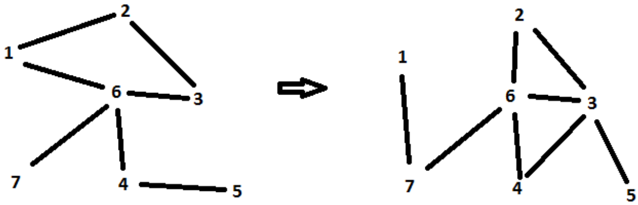
Theorem (Crane 2016)

Let $\Gamma = (\Gamma_t)_{t\geq 0}$ be an exchangeable càdlàg Markov process on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Then there are three types of discontinuity:

(I) **global jump**: *a positive fraction of all edges changes;*

(II) **single-vertex jump**: *a positive fraction of edges incident to a single vertex change, everything else stays the same;*

(III) **single-edge flip**: *a single edge changes, everything else stays the same.*



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$\Gamma = (\Gamma_t)_{t \geq 0}$ *an exchangeable càdlàg Markov process on* $\{0,1\}^{\mathbb{N} \times \mathbb{N}}$. *Then there are three types of discontinuity:*

(I) **global jump***: a positive fraction of all edges changes;*

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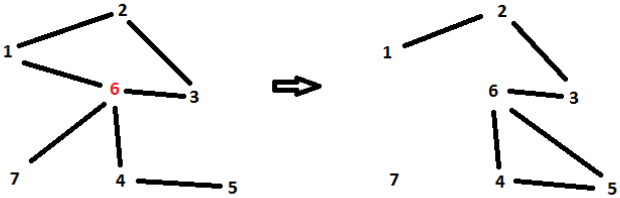

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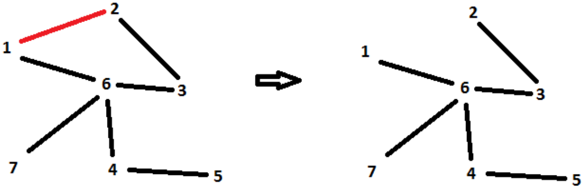
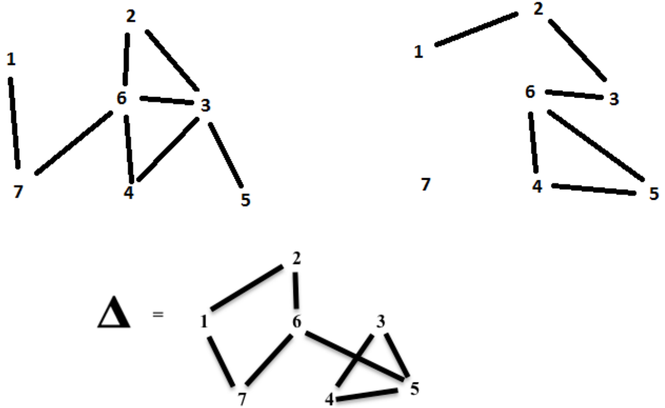


Combinatorial Lévy Processes

Define the **increment between graphs** G **and** G' **in** $\{0,1\}^{n\times n}$ $(n = 1, 2, \ldots, \infty)$ by $\Delta(G, G') \equiv (\Delta_{G,G'}(i,j))_{1\leq i,j\leq n}$, where

$$\Delta_{G,G'}(i,j) = |G_{ij} - G'_{ij}| = \begin{cases} 0, & G_{ij} = G'_{ij}, \\ 1, & \text{otherwise.} \end{cases}$$



Combinatorial Lévy processes (CLPs)

Define the **increment between graphs G and G' in $\{0,1\}^{n\times n}$** ($n = 1, 2, \ldots, \infty$) by $\Delta(G, G') \equiv (\Delta_{G,G'}(i,j))_{1\leq i,j\leq n}$, where

$$\Delta_{G,G'}(i,j) = |G_{ij} - G'_{ij}| = \begin{cases} 0, & G_{ij} = G'_{ij}, \\ 1, & \text{otherwise.} \end{cases}$$

Definition (CLP)

A **graph-valued Lévy process** $\Gamma = (\Gamma_t)_{t\geq 0}$ *on* $\{0,1\}^{n\times n}$ *($n = 1, 2, \ldots, \infty$) satisfies*

- $\Gamma_0 = \emptyset$ *(the empty graph),*
- stationary increments*: for all $s, t \geq 0$, the increment $\Delta(\Gamma_s, \Gamma_{s+t}) =_{\mathcal{D}} \Gamma_t$.*
- independent increments*: for all $t_0 < t_1 < \cdots < t_n < \infty$, the increments $\Delta(\Gamma_{t_1}, \Gamma_{t_0}), \ldots, \Delta(\Gamma_{t_n}, \Gamma_{t_{n-1}})$ are independent.*
- càdlàg paths*: Γ has càdlàg paths in the product-discrete topology on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$.*

Graph-valued random walk

Let μ be a probability distribution on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ and define $\Gamma = (\Gamma_t)_{t=0,1,\ldots}$ with $\Gamma_0 = G$ by

$$\Gamma_{t+1} = \Gamma_t \triangle D_t = \Gamma_{t-1} \triangle D_{t-1} \triangle D_t = G \triangle D_0 \triangle \cdots \triangle D_t,$$

for $D_0, D_1, \ldots$ i.i.d. μ.

Theorem

Any discrete time graph-valued Lévy process is uniquely determined by its initial distribution and the increment measure μ given above.

Open Problem

Study the convergence rate of CLPs and/or rewiring chains to their stationary distribution. Prove the cutoff phenomenon.

See related work:

> *H. Crane and S.P. Lalley. (2013). Convergence rates of Markov chains on spaces of partitions.* Electronic Journal of Probability.

Continuous-time CLPs (Poissonian construction)

Let μ be a measure on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ such that

$$\mu(\{\mathbf{0}_{\mathbb{N}}\}) = 0 \quad \text{and} \quad \mu(\{G \in \{0,1\}^{\mathbb{N}\times\mathbb{N}} : G|_{[n]} \neq \mathbf{0}_{[n]}\}) < \infty \quad \text{all } n \geq 1.$$

Let $\mathbf{D} := \{(t, D_t)\}$ be a Poisson point process on $(0,\infty) \times \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ with intensity $dt \otimes \mu$.

Construct $\Gamma^* = (\Gamma^*_t)_{t\geq 0}$ on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by putting $\Gamma^{*[n]} = \mathbf{0}_{[n]}$ for each $n \geq 1$ and

- $\Gamma^{*[n]}_t = \Gamma^{*[n]}_{t-} \triangle D_t$ for all atoms (t, D_t) in $\mathbf{D}$ for which $D_t|_{[n]} \neq \mathbf{0}_{[n]}$.
- $\Gamma^{*[n]}_t = \Gamma^{*[n]}_{t-}$ otherwise.

Theorem (Crane, 2018)

If μ is exchangeable, then it decomposes as

$$\mu = \mu_{(2)} + \mu_{(1,1)} + \mu_{(1)} + \mu_{\emptyset},$$

where

- $\mu_{(2)}$ *governs jumps of a single loop.*
- $\mu_{(1,1)}$ *governs jumps of a single edge (off-diagonal).*
- $\mu_{(1)}$ *governs jumps of a single vertex.*
- $\mu_{\emptyset}$ *governs global jumps.*

Induced process on graph limits

Theorem (C. 2016; C. 2017)

Let Γ *be an exchangeable Markov process on* $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ **(with the Feller property)***. Then the projection* $\|\Gamma\| := (\|\Gamma_t\|)_{t\geq 0}$ *into the space of graph limits* $\mathcal{D}^*$ *exists a.s.* **(and has the Feller property)** *with respect to the topology induced by*

$$d(D, D') := 2^{-n}\sum_{n\geq 1}\sum_{F,F'\in\{0,1\}^{n\times n}} |D(F) - D(F')|, \quad D, D' \in \mathcal{D}^*.$$

Open Problem

Construct and study an analog to "Brownian motion" in the space of graph limits.

- *S. Athreya, F. den Hollander and A. Röllin. (2019). Graphon-valued stochastic processes from population genetics. arXiv:1908.06241.*

Open Problem

Study projection of graph-valued Lévy processes into the space of graph limits.

- *Is there a suitable notion of Lévy process in this space?*
- *How does it relate to graph-valued Lévy processes?*
- *Address the problem for general combinatorial Lévy processes.*

$(\Gamma_t)_{t\geq 0}$ is a Markov process on $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ satisfying

- exchangeability*: for all $\sigma : \mathbb{N} \to \mathbb{N}$, the transition probabilities satisfy*

$$\mathbb{P}\{\Gamma_{t+1} \in \cdot \mid \Gamma_t = G\} = \mathbb{P}\{\Gamma^{\sigma}_{t+1} \in \cdot \mid \Gamma^{\sigma}_t = G\}, \quad \textit{for all } G \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}.$$

- projective Markov property*: $(\Gamma_t|_{[n]})_{t\geq 0}$ is a Markov chain on $\{0,1\}^{n\times n}$, for every $n = 1, 2, \ldots$.*

Theorem (Discrete-time processes, C. 2017)

Then there exists an exchangeable probability measure ω on $\{0,1\} \times \{0,1\}$-valued arrays so that $(\Gamma_m) =_{\mathcal{D}} (\Gamma^*_m)$ with $\Gamma^*_0 =_{\mathcal{D}} \Gamma_0$ and for each $m \geq 1$

$$\Gamma^*_m = W_m(\Gamma^*_{m-1}) = (W_m \circ \cdots \circ W_1)(\Gamma^*_0), \quad m \geq 1,$$

for $W_1, W_2, \ldots$ i.i.d. ω and $G' = W(G)$ defined by

$$G'_{ij} = \begin{cases} W^0_{ij}, & G_{ij} = 0, \\ W^1_{ij}, & G_{ij} = 1. \end{cases}$$

Summary

$(\Gamma_t)_{t\geq 0}$ *is a Markov process on* $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ *satisfying*

- exchangeability*: for all* $\sigma : \mathbb{N} \to \mathbb{N}$*, the transition probabilities satisfy*

$$\mathbb{P}\{\Gamma_{t+s} \in \cdot \mid \Gamma_t = G\} = \mathbb{P}\{\Gamma_{t+s}^{\sigma} \in \cdot \mid \Gamma_t^{\sigma} = G\}, \quad \textit{for all } G \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}.$$

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Theorem (Continuous-time processes, C. 2017)

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such that $\Gamma =_{\mathcal{D}} \Gamma_{\omega}^*$*, where* Γ_{ω}^* *is constructed from a Poisson point process with intensity* $dt \otimes \omega$*.*

Summary

$(\Gamma_t)_{t\geq 0}$ *is a Markov process on* $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ *satisfying*

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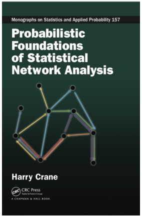

* More information available at **www.harrycrane.com/networks.html**